# Derivation of the Dual-Expression for Earth-Mover's (Wasserstein-1) Distance

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## 1 Introduction

Given two pdfs $f(\cdot)$, $g(\cdot)$, on $\mathbb{R}^n$, the earth-mover's distance between them is defined as

$$W(f,g) = \inf_{T \in \Pi(f,g)} \left\{ \mathbb{E}_T(\|x-y\|) = \iint \|x-y\| \, T(x,y) \, dx \, dy \right\}, \tag{1}$$

where $\Pi(f,g)$ denotes the set of pdfs on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals $f$ and $g$. That is, if $T \in \Pi(f,g)$, then it satisfies,

$$\int T(x,y) \, dy = f(x), \tag{2}$$

$$\int T(x,y) \, dx = g(y). \tag{3}$$

In the following, we will show the well-known results that this distance can also be computed as

$$W(f,g) = \sup_{h \in \text{1-Lip}} \left\{ \mathbb{E}_f(h(\cdot) - \mathbb{E}_g(h(\cdot) = \int f(x) \, h(x) dx - \int g(x) \, h(x) dx \right\}, \tag{4}$$

where 1-Lip denotes the set of functions that satisfy

$$\|h(x) - h(y)\| \leq \|x - y\|. \tag{5}$$

## 2 The Dual Problem

This result follows by considering the dual problem obtained via the Lagrangian. We first form the equivalent saddle problem for (1) as

$$\begin{aligned} \inf_T \sup_{\lambda_0(\cdot), \lambda_1(\cdot), \beta(\cdot,\cdot) \geq 0} & \iint \|x-y\| \, T(x,y) \, dx \, dy \\ & - \int \lambda_0(x) \left[ \int T(x,y) \, dy - f(x) \right] dx - \int \lambda_1(y) \left[ \int T(x,y) \, dx - g(y) \right] dy \\ & - \int \beta(x,y) \, T(x,y) \, dx \, dy. \end{aligned} \tag{6}$$

Note that here, $\lambda_i(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and $\beta(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Rearranging, we can rewrite the saddle-point problem as

$$\sup_{\lambda_0(\cdot),\lambda_1(\cdot),\beta(\cdot,\cdot)\geq 0} \inf_T \int \lambda_0(x)\, f(x)\, dx + \int \lambda_1(x)\, g(x)\, dx \\ + \iint T(x,y)\left[\|x-y\| - \lambda_0(x) - \lambda_1(y) - \beta(x,y)\right] dx\, dy. \quad (7)$$

Note that for the inner minimization over $T$ to assume a finite value, the terms inside the square bracket must evaluate to zero. Equivalently, we must have

$$\lambda_0(x) + \lambda_1(y) + \beta(x,y) = \|x-y\|. \quad (8)$$

Because of the constraint $\beta(x,y) \geq 0$, this implies that

$$\lambda_0(x) + \lambda_1(y) \leq \|x-y\|. \quad (9)$$

Evaluating this at $y = x$, we find

$$\lambda_0(x) + \lambda_1(x) \leq 0 \quad \Longleftrightarrow \quad \lambda_1(x) \leq -\lambda_0(x). \quad (10)$$

Let us use these observations in (7). Note that, since the term that involves $T$ evaluates to zero, (7) reduces to

$$\sup_{\lambda_0(\cdot),\lambda_1(\cdot)} \int \lambda_0(x)\, f(x)\, dx + \int \lambda_1(x)\, g(x)\, dx \quad \text{subject to } \lambda_0(x) + \lambda_1(y) \leq \|x-y\| \quad (11)$$

Inserting the inequality in (10), we find that the term to be maximized is upper bounded by

$$\int \lambda_0(x)\, f(x)\, dx - \int \lambda_0(x)\, g(x)\, dx. \quad (12)$$

However, this upper bound can be satisfied by setting $\lambda_1(x) = -\lambda_0(x)$. Plugging this in (and renaming $\lambda_0$ as $h$, the maximization problem reduces to

$$\sup_{h(\cdot)} \int h(x)\, f(x)\, dx - \int h(x)\, g(x)\, dx \quad \text{subject to } h(x) - h(y) \leq \|x-y\|. \quad (13)$$

This is exactly the problem in (4), (5).