# EHB 252E – Signals and Systems

Spring 2012

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Class Meets : 13.30 – 16.30, Thursday
EEB 5204

Office Hours : 14.00 – 16.00, Tuesday

Textbook : A. V. Oppenheim, A. S. Willsky, S. H. Nawab, 'Signals and Systems', 2nd Edition, Prentice Hall.

Grading : Homeworks and Quizzes (10%), 2 Midterms (25% each), Final (40%).

**Tentative Course Outline**

- Introduction to Signals and Systems

  *Basic discrete and continuous-time signals and their properties, transformations of the independent variable, basic system properties*

- Linear Time-Invariant Systems

  *Discrete and continuous time LTI systems, convolution sum and integral, properties of LTI systems, LTI systems defined by differential and difference equations*

- Fourier Series

  *Representation of continuous and discrete-time signals using Fourier series, properties of Fourier series, LTI systems and Fourier series*

- The Continuous-Time Fourier Transform

- The Discrete-Time Fourier Transform

- Frequency Domain Characterization of Systems

  *Magnitude-phase representations, ideal frequency selective filters*

- Sampling

  *A/D, D/A conversion, aliasing, discrete-time processing of continuous-time signals*

- The $z$-Transform

  *The $z$ transform, region of convergence, inverse $z$-transform, properties of the $z$-transform, characterizing LTI systems using the $z$-transform*

- The Laplace Transform

  *The Laplace transform, region of convergence, inverse Laplace transform, properties of the Laplace transform, characterizing LTI systems using the Laplace transform*

# EHB 252E – Homework 1

Due 23.02.2012

1. Let $S_1$, $S_2$ be continous-time systems whose input-output relations are as specified below.

$$x(t) \longrightarrow \boxed{S_1} \longrightarrow x\big(t/3\big) \qquad\qquad x(t) \longrightarrow \boxed{S_2} \longrightarrow x\big(t-1\big)$$

Also, let $f(t)$ be a signal described as

$$f(t) = \begin{cases} (t-2)^2, & \text{for } 1 \le t < 2 \\ 2-t, & \text{for } 2 \le t \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch $f(t)$.

(b) Consider the system below.

$$x(t) \longrightarrow \boxed{S_1} \longrightarrow \boxed{S_2} \longrightarrow y(t)$$

For $x(t) = f(t)$, determine and sketch $y(t)$.

(c) Consider the system below.

$$x(t) \longrightarrow \boxed{S_2} \longrightarrow \boxed{S_1} \longrightarrow z(t)$$

For $x(t) = f(t)$, determine and sketch $z(t)$.

2. Let $S$ be a <u>time-invariant</u> system that maps a discrete-time signal $x$ to a discrete-time signal $y$. Suppose we know that

$$y(0) = 3\,x(0) - 2\,x(-1) + x(-2).$$

(a) Determine an expression for $y(1)$ in terms of the samples of $x$.

(b) Determine an expression for $y(n)$ in terms of the samples of $x$.

(c) Is $S$ linear or not? Please (briefly) explain your answer. If information is insufficient, explain why you think so.

3. We repeat the question above for a continuous-time system. This time, let $S$ be a time-invariant system that maps a continuous-time signal $x$ to a continuous-time signal $y$. Suppose we know that

$$y(0) = \int_{-1}^{2} x(t)\, f(t)\, dt,$$

where $f(t)$ is a continuous-time signal.

(a) Determine an expression for $y(1)$ in terms of $x$ and $f$.

(b) Determine an expression for $y(t)$ in terms of $x$ and $f$.

(c) Is $S$ linear or not? Please (briefly) explain your answer. If information is insufficient, explain why you think so.

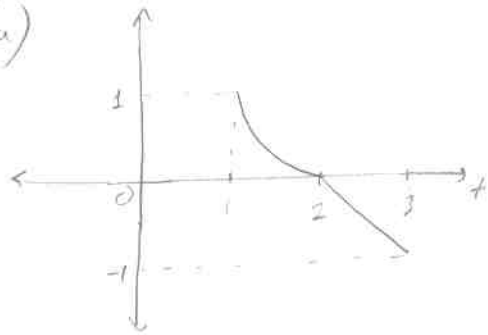
4. Let $x$ and $y$ be discrete-time signals defined as, $x(n) = e^{-n^2}\sin(n)$, $y(n) = e^{-n^2}\cos(n)$. Also let $z = x * y$. Compute $\sum_{n=-\infty}^{\infty} z(n)$.
Hint : Try to express $\sum_n z(n)$ in terms of $\sum_n x(n)$ and $\sum_n y(n)$.

5. (a) Let $x$ and $y$ be discrete-time signals, described as $x(n) = u(n)$, $y(n) = e^{-n}\,u(n)$. Also, let $z(n) = x(n) * y(n)$. Compute $z(n)$.

(b) Let $x$ and $y$ be continuous-time signals described as $x(t) = u(t) - u(t-3)$, $y(t) = u(t) - u(t-1)$. Also, let $z(t) = x(t) * y(t)$. Compute and sketch $z(t)$.

EHB 252E - HW1 Solutions

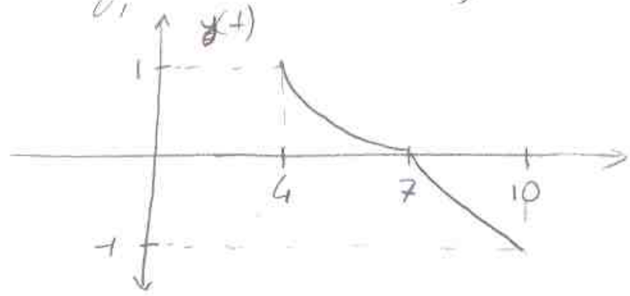
1) (a)




(b) $x \rightarrow \boxed{S_1} \xrightarrow{y_1} \boxed{S_2} \rightarrow y$

$y_1(t) = x(t/3)$

$y(t) = y_1(t-1) = x\left(\frac{t-1}{3}\right)$

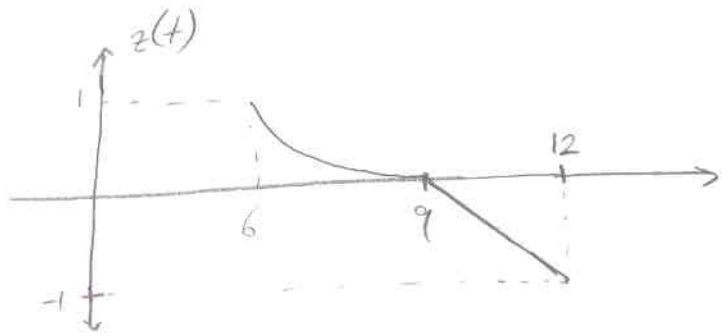



(c) $x \rightarrow \boxed{S_2} \xrightarrow{z_1} \boxed{S_1} \rightarrow z$

$z_1(t) = x(t-1)$

$z(t) = z_1(t/3) = x(t/3 - 1)$

$\left(= y(t-2)\right)$




2.) (a) Suppose we input $\tilde{x}(n) = x(n+1)$. Then the output $\tilde{y}(n)$ satisfies $\tilde{y}(0) = 3\tilde{x}(0) - 2\tilde{x}(-1) + \tilde{x}(-2) = 3x(1) - 2x(0) + x(-1)$ $(*)$

But, by time-invariance, we should have $\tilde{y}(n) = y(n+1)$, so $\tilde{y}(0) = y(1)$, so $(*)$ is the desired expression.

(b) This time input $\tilde{x}(n) = x(n+k)$. By the same argument, we obtain, $\tilde{y}(n) = y(n+k)$ and, $\tilde{y}(0) = y(k) = 3x(k) - 2x(k-1) + x(k-2)$

(c) The expression $y(n) = 3x(n) - 2x(n-1) + x(n-2)$ is actually a convolution sum $\Rightarrow$ The system is LTI.

(3) (a) Let $\tilde{x}(t) = x(t+1)$ be the input and $\tilde{y}(t)$ the output.

By time-invariance $\tilde{y}(t) = y(t+1)$

$$\tilde{y}(0) = y(1) = \int_{-1}^{2} \tilde{x}(t) f(t)\,dt = \int_{-1}^{2} x(t+1) f(t)\,dt$$

(b) Similarly, let $\tilde{x}(t) = x(t+s)$ be the input and $\tilde{y}(t)$ be the output. By t-i we have $\tilde{y}(t) = y(t+s)$.

$$\tilde{y}(0) = y(s) = \int_{-1}^{2} \tilde{x}(t) f(t)\,dt = \int_{-1}^{2} x(t+s) f(t)\,dt$$

(c) If we define $h(t) = f(-t)$, we can write (for $\tau = -s$)

$$y(t) = \int_{-1}^{2} x(s+t) f(s)\,ds = \int_{-2}^{1} x(t-\tau) f(-\tau)\,d\tau$$

$$= \int_{-2}^{1} x(t-\tau) h(\tau)\,d\tau$$

This is a convolution integral $\Rightarrow$ system is LTI.

(4) We have $z(n) = \sum_k x(n-k) y(k)$. If we sum over $n$,

$$\sum_{n=-\infty}^{\infty} z(n) = \sum_{n=-\infty}^{\infty} \sum_k x(n-k) y(k) = \sum_k y(k) \sum_{n=-\infty}^{\infty} x(n-k)$$

Notice that $\sum_{n=-\infty}^{\infty} x(n-k) = \sum_{n=-\infty}^{\infty} x(n)$ for any value of $k$.

since $x$ is odd.

Thus, $$\sum_{n=-\infty}^{\infty} z(n) = \left[\sum_{n=-\infty}^{\infty} y(n)\right] \left[\sum_{n=-\infty}^{\infty} x(n)\right] = \left[\sum_n y(n)\right] \cdot 0 = 0$$

(5.) (a) $z(n) = \sum_{k=-\infty}^{\infty} u(n-k)\, e^{-k} \cdot u(k)$

Notice that $u(n-k) \cdot u(k) = \begin{cases} 1 & \text{for } 0 \le k \le n. \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \quad z(n) = \sum_{k=0}^{n} e^{-k} = \frac{e^{-(n+1)} - 1}{e^{-1} - 1}$$

# EHB 252E – Homework 2

Due 08.03.2012

1. Let $x_1(t) = u(t) - u(t-1)$, $x_2(t) = u(t) - u(t-2)$, $x_3(t) = e^{-2\,t}\,u(t)$.

   (a) Let $y_1 = x_1 * x_1$. Determine and sketch $y_1(t)$.

   (b) Let $y_2 = x_1 * x_2$. Determine and sketch $y_2(t)$.

   (c) Let $y_3 = x_1 * x_3$. Determine and sketch $y_3(t)$.

   (d) Let $y_4 = x_3 * x_3$. Determine and sketch $y_4(t)$.

2. Consider a discrete-time system $S$ desribed by the difference equation

$$y(n) - y(n-1) + \frac{1}{4}\,y(n-2) = x(n), \tag{1}$$

   and 'initial rest' conditions. In this question, we will derive the impulse response of this LTI system, by viewing it as a cascade of two first-order systems.

   (a) Consider the LTI system $S_\alpha$ described by the difference equation $y(n) = \alpha\,y(n-1) + x(n)$ and initial rest conditions. Derive the impulse response of $S_a$. Notice that the impulse response will depend on the variable $\alpha$.

   (b) Consider the cascade system

   $$\longrightarrow \boxed{S_{\alpha_1}} \longrightarrow \boxed{S_{\alpha_2}} \longrightarrow$$

   Express the difference equation associated with this system.

   (c) Determine $\alpha_1$, $\alpha_2$, so that the difference equation is equivalent to (**??**) above.

   (d) Determine the impulse response of $S$.

3. Let $S_1$ and $S_2$ be LTI systems. Also, let $S$ denote the cascade system below.

   $$\longrightarrow \boxed{S_1} \longrightarrow \boxed{S_2} \longrightarrow$$

   Suppose that

   - the impulse response of $S_2$ is $h_2(n) = \delta(n) - \frac{1}{2}\,\delta(n-1)$,
   - the impulse response of $S$ is $h(n) = 2\,\delta(n) + 3\,\delta(n-1) - \delta(n-3)$.

   Determine $h_1(n)$, the impulse response of $S_1$.

4. Consider an LTI system whose input $x(t)$ and output $y(t)$ satisfy

$$y(t) = \int_{-\infty}^{\infty} 3\,x(\tau+2)\,f(2t+2\tau)\,d\tau.$$

   where

$$f(t) = \begin{cases} 1 & \text{for } 1 \le t \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

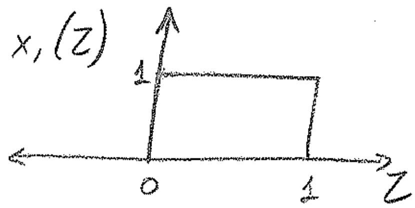
   Determine and sketch the impulse response of the system.

5. Let $x(t) = \cos(4\pi\,t) + \sin(4\pi\,t)$. Determine the Fourier series coefficients of $x(t)$. That is, find $a_k$ so that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\,e^{j\,2\pi k\,t}.$$

EHB252E - HW2 Solutions

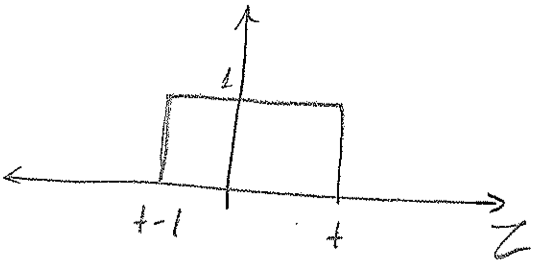
1. (a) Notice $x_1(\tau)$




$\Rightarrow$ Let $\tilde{x}_1(\tau) = x_1(-\tau)$

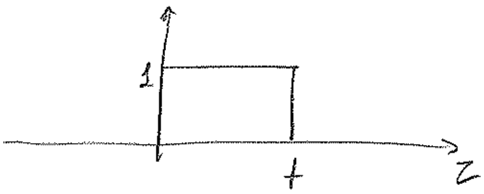
Then, $\tilde{x}_1(\tau - t) = x_1(t-\tau)$

So, $x_1(t-\tau)$ (as a function of $\tau$)

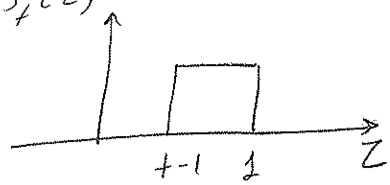



$\Rightarrow s_t(\tau) = x_1(\tau)\cdot x_1(t-\tau)$ is non-zero only for $0 \le t \le 2$

If $0 \le t < 1$, $s_t(\tau)$

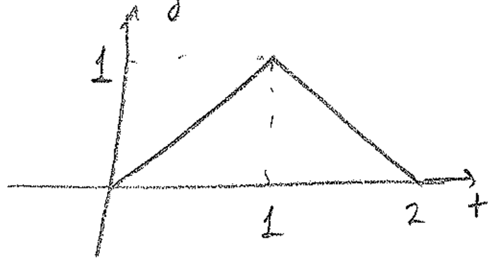



If $1 \le t < 2$ $s_t(\tau)$



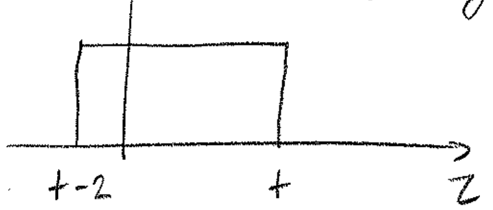

But $y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_1(t-\tau)\, d\tau = \int s_t(\tau)\, d\tau$

$$= \begin{cases} \int_0^t 1\cdot d\tau = t, & \text{if } 0 \le t < 1 \\ \int_{t-1}^{1} 1\cdot d\tau = 2-t, & \text{if } 1 \le t \le 2 \\ 0, & \text{if } t \notin [0,2] \end{cases}$$



(b) Similarly let $\tilde{x}_2(\tau) = x_2(-\tau)$. Then $\tilde{x}_2(\tau - t) = x_2(t-\tau)$

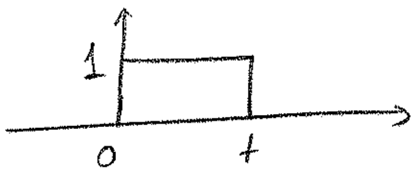
$x_2(t-\tau)$ (as a function of $\tau$)




$\Rightarrow s_t(\tau) = x_1(\tau)\, x_2(t-\tau)$ is non-zero for $0 \le t \le 3$.

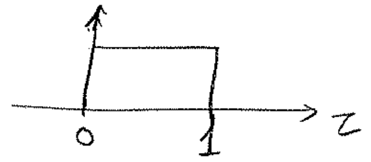
for $0 \le t \le 1$

$s_t(\tau)$



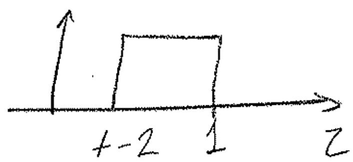

for $1 \le t \le 2$

$s_t(\tau)$

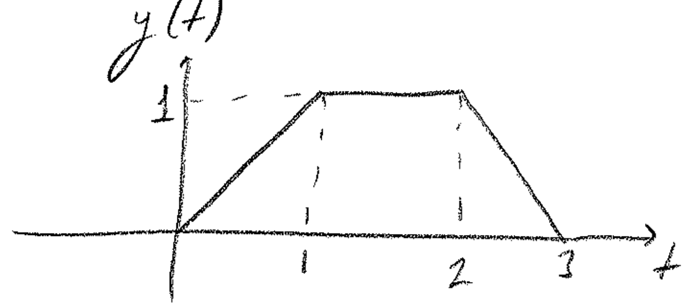



for $2 \le t \le 3$,

$s_t(\tau)$




$$\Rightarrow y(t) = \int_{-\infty}^{\infty} s_t(\tau)\, d\tau = \begin{cases} \int_0^t 1\, d\tau = t, & \text{if } 0 \le t \le 1 \\ \int_0^1 1 \cdot d\tau = 1, & \text{if } 1 \le t \le 2 \\ \int_{t-2}^1 1 \cdot d\tau = 3-t, & \text{if } 2 \le t \le 3 \\ 0 & \text{if } t \notin [0,3] \end{cases}$$

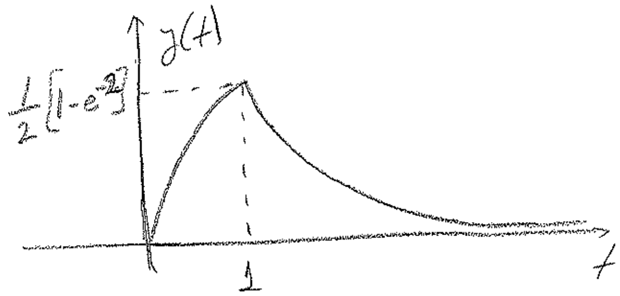


(c) Notice that $s_t(\tau) = x_3(\tau) \cdot x_1(t-\tau) = 0$ if $t < 0$

for $1 > t > 0$, $s_t(\tau) = \begin{cases} e^{-2\tau} & \text{for } 0 < \tau < t, \\ 0 & \text{for } \tau > t \end{cases}$

for $t > 1$, $s_t(\tau) = \begin{cases} e^{-2\tau}, & \text{for } t-1 < \tau < t \\ 0, & \text{otherwise} \end{cases}$

Therefore, $y(t) = \int_{-\infty}^{\infty} s_t(\tau)\, d\tau = \begin{cases} 0, & \text{for } t < 0. \\ \int_0^t e^{-2\tau} d\tau = \frac{1}{2}\left[1 - e^{-2t}\right] & \text{for } 0 < t < 1. \\ \int_{t-1}^{t} e^{-2\tau} d\tau = \frac{1}{2} e^{-2t}\left[e^2 - 1\right] & \text{for } 1 \le t. \end{cases}$



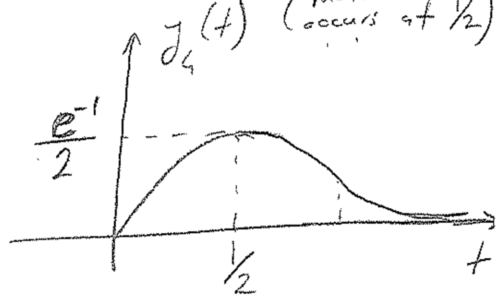

(d) $y_4(t) = \int_{-\infty}^{\infty} x_3(\tau)\, x_3(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau) \cdot e^{-2(t-\tau)} u(t-\tau)\, d\tau$

Notice that if $t < 0$, $u(\tau) \cdot u(t-\tau) = 0$ for all $\tau$.

if $t \ge 0$, $u(\tau) \cdot u(t-\tau) = \begin{cases} 1 & \text{for } 0 \le \tau \le t \\ 0, & \text{otherwise} \end{cases}$

Therefore

$y_4(t) = \begin{cases} 0, & \text{if } t < 0, \\ \int_0^t e^{-2t}\, d\tau = t \cdot e^{-2t}, & \text{if } t > 0 \end{cases}$



(2.) (a) If we set $x(n) = \delta(n)$, the initial rest conditions require that $y(-1) = 0$ (since $x(n) = 0$ for $n < 0$).

$$\Rightarrow y(0) = \alpha \cdot \overbrace{y(-1)}^{0} + \overbrace{\delta(0)}^{1} = 1$$

$$y(1) = \alpha \cdot \overbrace{y(0)}^{1} + \overbrace{\delta(1)}^{0} = \alpha \quad \Longrightarrow y(n) = \alpha^n \cdot u(n).$$

$$y(2) = \underbrace{\alpha \cdot \overbrace{y(1)}^{\alpha}}_{= \alpha^2 \cdot y(0)} + \overbrace{\delta(2)}^{0} = \alpha^2 \quad \Rightarrow \text{The impulse response is } h(n) = \alpha^n \cdot u(n)$$

$$y(n) = \alpha \cdot y(n-1) + \delta(n) = \alpha \cdot y(n-1) \text{ for } n > 0$$

(b) Let $x(n) \longrightarrow \boxed{S_{\alpha_1}} \xrightarrow{y_1(n)} \boxed{S_{\alpha_2}} \longrightarrow y(n).$

$$\Rightarrow y_1(n) = \alpha_1 y_1(n-1) + x(n) \Rightarrow y_1(n) - \alpha_1 \cdot y_1(n-1) = x(n) \quad (*)$$

$$y(n) = \alpha_2 \cdot y(n-1) + y_1(n) \quad (**)$$

To get rid of $y_1(n)$ in this eqn., notice that

$$\alpha_1 \cdot y(n-1) = \alpha_1 \cdot \alpha_2 \cdot y(n-2) + \alpha_1 \cdot y_1(n-1)$$

Subtract this from $(**)$ and use $(*)$ to obtain:

$$y(n) - \alpha_1 y(n-1) = \alpha_2 y(n-1) - \alpha_1 \alpha_2 y(n-2) + \underbrace{y_1(n) - \alpha_1 y_1(n-1)}_{= x(n)}$$

Rearranging we obtain:

$$y(n) - (\alpha_1 + \alpha_2) y(n-1) + \alpha_1 \alpha_2 y(n-2) = x(n).$$

(c) Comparing the difference equ. in (b) and the question statement, we see that they're equivalent if

$$\alpha_1 + \alpha_2 = 1$$
$$\alpha_1 \alpha_2 = 1/4$$

$$\Rightarrow \alpha_1 = \alpha_2 = \frac{1}{2}.$$

(d) The impulse response of $\rightarrow \boxed{S_1} \rightarrow \boxed{S_2} \rightarrow$ is given by $h_1(n) * h_2(n)$ where $h_1, h_2$ are the impulse responses of $S_1$ and $S_2$.

$$\Rightarrow h(n) = \left[\left(\frac{1}{2}\right)^n \cdot u(n)\right] * \left[\left(\frac{1}{2}\right)^n \cdot u(n)\right]$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \cdot u(k) \cdot \left(\frac{1}{2}\right)^{n-k} \cdot u(n-k)$$

(Note: $u(k) \cdot u(n-k) = \begin{cases} 1 & \text{if } 0 \le k \le n \\ 0 & \text{else} \end{cases}$)

$$= \begin{cases} 0, & \text{if } n < 0 \\ \displaystyle\sum_{k=0}^{n} \left(\frac{1}{2}\right)^n = (n+1) \cdot \left(\frac{1}{2}\right)^n & \text{if } n \ge 0. \end{cases}$$

3. $h(n) = h_1(n) * h_2(n) = h_1(n) * \left(\delta(n) - \frac{1}{2}\delta(n-1)\right)$

$$\Longrightarrow h(n) = h_1(n) - \frac{1}{2} h(n-1)$$

Note: There's missing information. Assume that $h_1(n)$ is 'causal' i.e. $h_1(n) = 0$ for $n < 0$.

Under this assumption,

$$\begin{array}{lllllll} & [\,h_1(0) & h_1(1) & h_1(2) & h_1(3) & \cdots\,] & \Rightarrow h_1(n) \\ -\frac{1}{2}h_1(-1) & [\,0 & -\frac{1}{2}h_1(0) & -\frac{1}{2}h_1(1) & -\frac{1}{2}h_1(2) & \cdots\,] & \Rightarrow -\frac{1}{2}h_1(n-1) \\ + & & & & & & \\ \hline & [\,h(0) & h(1) & h(2) & h(3) & \cdots\,] & \Rightarrow h(n) \\ = & [\,2 & 3 & 0 & -1 & \cdots\,] & \to 0 \end{array}$$

Therefore, $h_1(0) = 2$,

$$h_1(1) = h(1) + \frac{1}{2} h_1(0) = 4$$

$$h_1(2) = h(2) + \frac{1}{2} h_1(1) = 2$$

$$h_1(3) = h(3) + \frac{1}{2} h_1(2) = 0$$

$$h_1(4) = h(4) + \frac{1}{2} h_1(3) = 0$$

$$\vdots$$

$$h_1(n) = 0 \quad \text{for} \quad n \geq 4.$$

$$\Rightarrow h_1(n) = 2\delta(n) + 4\delta(n-1) + 2\delta(n-2)$$

<u>Note</u>: Without the 'causality' assumption, notice that " $h_1(n) + c\cdot\left(\frac{1}{2}\right)^n$ " is also a solution. Here, $c$ is an arbitrary constant. ($\left(\frac{1}{2}\right)^n$ is the 'homogeneous solution' of the difference eqn. $y(n) - \frac{1}{2}y(n-1) = 0$).

(4.) <u>Note</u>: There's a typo in the equation that relates $y(t)$ to $x(t)$. It should be:

$$y(t) = \int_{-\infty}^{\infty} 3\,x(\tau+2)\, f(2t - 2\tau)\, d\tau$$

a minus instead of a plus

(otherwise the system is not time invariant.)

In this case, if we set $\tau + 2 = t - s$,

$(2\tau = 2t - 2s - 4)$

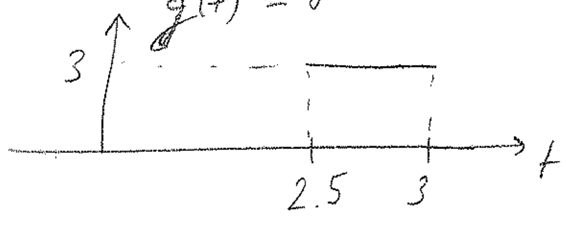
$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(t-s) \cdot 3 \cdot f(2s-4)\, ds$$

Now let $g(s) = 3 \cdot f(2s-4)$ then,

$$y(t) = \int_{-\infty}^{\infty} x(t-s)\, g(s)\, ds. \Rightarrow g(s) = 3 \cdot f(2s-4) \text{ is}$$

the impulse response.

$$g(t) = 3 \cdot f(2t-4)$$



5.

$$x(t) = \underbrace{\frac{1}{2}e^{j(2\pi)\cdot 2t} + \frac{1}{2}e^{-j(2\pi)\cdot 2t}}_{\cos 6\pi t} + \underbrace{\frac{1}{2j}e^{j(2\pi)\cdot 2t} - \frac{1}{2j}e^{-j(2\pi)2t}}_{\sin(4\pi t)}$$

$$= \left(\frac{1}{2} + \frac{1}{2j}\right) e^{j(2\pi)\cdot 2t} + \left(\frac{1}{2} - \frac{1}{2j}\right) e^{j(2\pi)(-2)t}$$

$$\Rightarrow a_2 = \frac{1}{2} + \frac{1}{2j}, \quad a_{-2} = \frac{1}{2} - \frac{1}{2j}$$

# EHB 252E – Homework 3

Due 05.04.2012

1. Let $u(t)$ denote the continuous-time unit step function.
   (a) Compute $u(t) * u(t)$.
   (b) Compute $u(t) * u(t) * u(t)$.
   (c) Compute $u(t) * \big(u(t) - u(t-1)\big)$. (Hint : Make use of part (a).)
   (d) Compute $u(t) * \left(\sum_{k=0}^{\infty} (-1)^k \, u(t-k)\right)$. (Hint : Make use of part (c).)

2. Let $u(n)$ denote the discrete-time unit step function.
   (a) Compute $u(n) * u(n)$.
   (b) Compute $u(n) * u(n) * u(n)$.
   (c) Let $f(n) = u(n) * u(n) * u(n) * u(n)$. Compute $d(n) = f(n) - f(n-1)$.

3. Let $f(t)$ be a periodic continuous-time signal with period 2. On $[-1, 1)$, $f(t)$ is described as,

$$f(t) = \begin{cases} 0, & \text{for } t \in [-1, -1/4), \\ 1, & \text{for } t \in [-1/4, 1/4], \\ 0, & \text{for } t \in (1/4, 1). \end{cases}$$

   Let us also define the 'sinc' function as,

$$\operatorname{sinc}(t) = \begin{cases} 1, & \text{for } t = 0, \\ \frac{\sin(\pi\, t)}{\pi\, t}, & \text{for } t \neq 0. \end{cases}$$

   Notice that, defined this way, 'sinc' is continuous.

   (a) Compute the Fourier series coefficients $a_k$ so that $f(t) = \sum_k a_k \, e^{j\pi\, k\, t}$. Express $a_k$ in terms of the sinc function.
   (b) Define a new sequence $b_k = a_k^2$. Let $g(t) = \sum_k b_k \, e^{j\pi\, k\, t}$. Determine and sketch $g(t)$. (Hint : Make use of the Fourier series properties.)

4. Let $S$ be a continuous-time LTI system with impulse response $h(t)$. Recall that if we input $x(t) = e^{j\omega\, t}$ to the system, the output is $y(t) = H(j\omega)\, e^{j\omega\, t}$ where $H(j\omega) = \int_{t=-\infty}^{\infty} h(t)\, e^{-j\omega\, t}\, dt$.
   Suppose that the input and the output are related by the differential equation

$$y(t) - \frac{3}{4}\, y'(t) = x(t) - x(t-1).$$

   (a) Let us input $x(t) = e^{j\omega\, t}$ to the system. Determine the output to obtain $H(j\omega)$.
   (b) Determine the impulse response $h(t)$. (Hint : Recall the Fourier transform of the function $e^{-st}\, u(t)$ for $s > 0$. Use the properties of the Fourier transform.)

EHB 252E - HW3 Solutions

1. (a) Let $r(t) = u(t) * u(t)$.

$$r(t) = \int_{-\infty}^{\infty} u(t-\tau)\cdot u(\tau)\, d\tau = \begin{cases} \int_0^t 1\, d\tau = t, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

$\Rightarrow r(t) = t\cdot u(t)$ (this is called the 'ramp' function)

(b) Let $y(t) = u(t) * u(t) * u(t)$

Since $u(t) * u(t) = r(t)$ and convolution is associative, we have

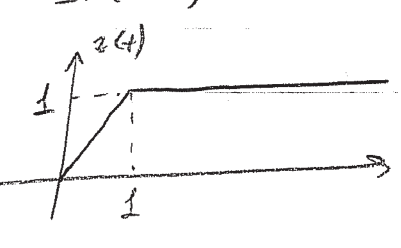
$$y(t) = r(t) * u(t)$$

$$= \int_{-\infty}^{\infty} \tau\cdot u(\tau)\cdot u(t-\tau)\, d\tau = \begin{cases} \int_0^t \tau\, d\tau = \frac{t^2}{2}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

(c) Let $z(t) = u(t) * [u(t) - u(t-1)]$

Then, $z(t) = \underbrace{u(t) * u(t)}_{= r(t)} - \underbrace{u(t) * u(t-1)}_{= r(t-1)}$

$$\Rightarrow z(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 1 & \text{for } t \geq 1 \\ 0 & \text{for } t < 0 \end{cases}$$



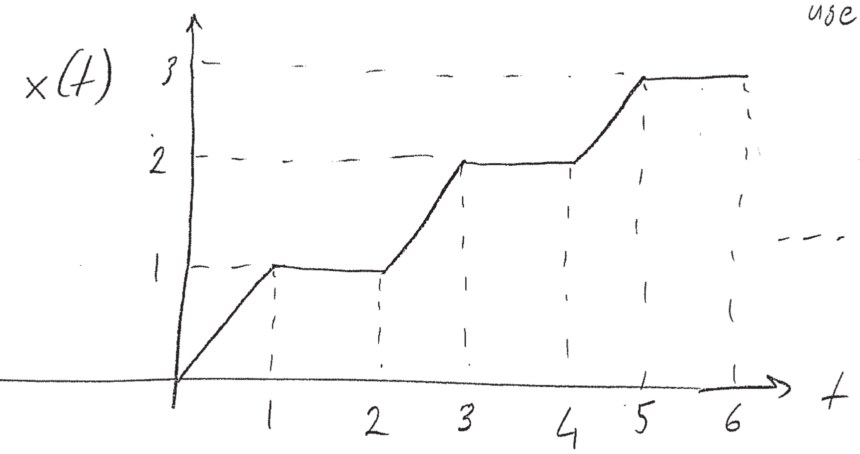
(d) Let $x(t) = u(t) * \left[ \sum_{k=0}^{\infty} (-1)^k \cdot u(t-k) \right]$

Then for $z(t) = u(t) * (u(t) - u(t-1))$, we can write

$x(t)$ as $x(t) = \underbrace{u(t) * [u(t) - u(t-1)]}_{z(t)} + \underbrace{u(t) * [u(t-2) - u(t-3)]}_{z(t-2)} + \ldots$

$$= \sum_{m=0}^{\infty} z(t-2m)$$

$x(t)$ can be expressed by making use of the "floor" function...

$\Rightarrow$ 


(2.) (a) Let $r(n) = u(n) * u(n)$.

$$r(n) = \sum_{k=-\infty}^{\infty} u(n-k) \cdot u(k) = \begin{cases} \sum_{k=0}^{n} 1 = n+1, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases}$$

$r(n)$: discrete ramp

(b) Let $y(n) = u(n) * u(n) * u(n)$. Then, $y(n) = r(n) * u(n)$.

$$\Rightarrow y(n) = \sum_{k=-\infty}^{\infty} r(k) \cdot u(n-k) = \begin{cases} \sum_{k=0}^{n} k = \frac{k \cdot (k+1)}{2} & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases}$$

② (c) $f(n) - f(n-1) = y(n) * u(n) - y(n) * u(n-1)$

from (b) $= y(n) * \underbrace{\left[u(n) - u(n-1)\right]}_{=\delta(n)} = y(n).$

③ (a) for $k \neq 0$,

$$a_k = \frac{1}{2} \int_{-1}^{1} f(t)\, e^{-j\pi k t}\, dt = \frac{1}{2} \int_{-1/4}^{1/4} e^{-j\pi k t}\, dt$$

$$= \frac{1}{2} \cdot \frac{1}{-j\pi k} \cdot e^{-j\pi k t} \Big|_{-1/4}^{1/4} = \frac{\sin \pi k/4}{\pi k}$$

$$= \frac{\sin\left(\frac{\pi k}{4}\right)}{\left(\frac{\pi k}{4}\right)} \cdot \frac{1}{4} = \frac{1}{4} \operatorname{sinc}\left(\frac{k}{4}\right)$$

For $k = 0$, $a_0 = \frac{1}{2} \int_{-1/4}^{1/4} 1 \cdot dt = \frac{1}{4} = \frac{1}{4} \cdot \operatorname{sinc}(0).$

---

$\Rightarrow a_k = \frac{1}{4} \operatorname{sinc}\left(\frac{k}{4}\right)$ for all $k \in \mathbb{Z}$.

(b) Recall that, if $x(t)$ and $y(t)$ are periodic with $T$ and

$\left.\begin{array}{l} x(t) \leftrightarrow c_k \\ y(t) \leftrightarrow d_k \end{array}\right\}$ then, for $z(t) = \frac{1}{T} \int_{s}^{s+T} x(\tau)\, y(t-\tau)\, d\tau$

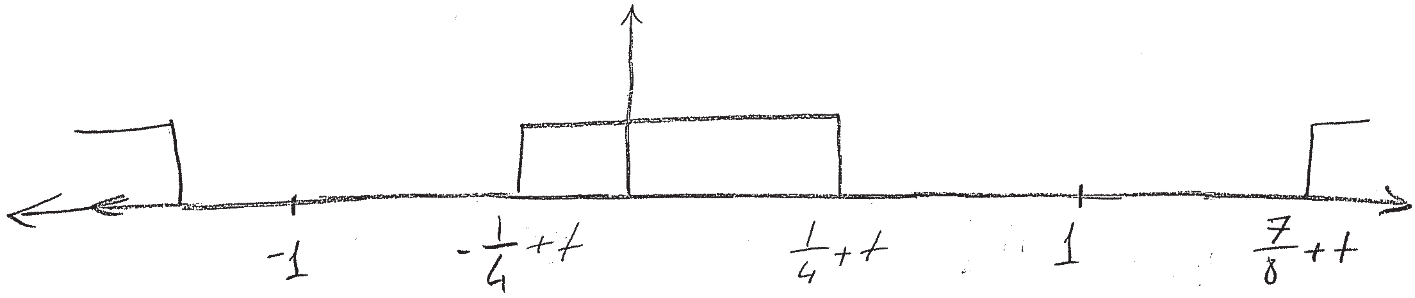
($s$ is arbitrary)

we have $z(t) \longleftrightarrow c_k \cdot d_k$

so, for $g(t) = \frac{1}{2} \int_{-1}^{1} f(\tau) \cdot f(t-\tau)\, d\tau$

$g(t) \longleftrightarrow a_k^2$.

As a function of $\tau$, $f(t-\tau)$ looks like (periodic with $T=2$)




$\Rightarrow$ for $-1 \le t \le -\frac{1}{2}$, $g(t) = \frac{1}{2} \int \underbrace{f(\tau) f(t-\tau)}_{=0} d\tau = 0$

for $-\frac{1}{2} \le t \le 0$, $g(t) = \frac{1}{2} \int_{-\frac{1}{4}}^{1/4+t} 1\, d\tau = \left(\frac{1}{2}+t\right)/2$

for $0 \le t \le \frac{1}{2}$, $g(t) = \frac{1}{2} \int_{-\frac{1}{4}+t}^{1/4} 1\, d\tau = \left(\frac{1}{2}-t\right)/2$

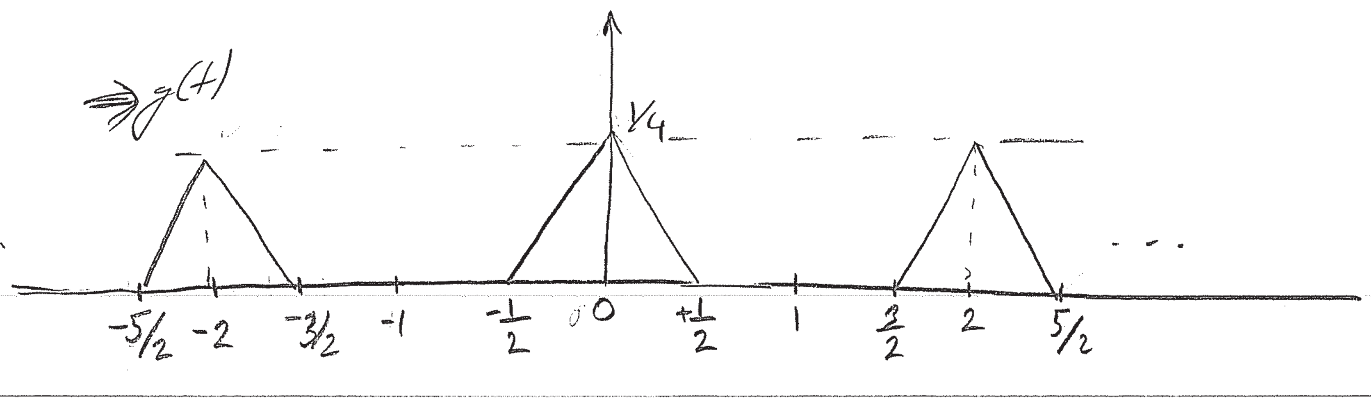
for $\frac{1}{2} \le t \le 1$, $g(t) = \frac{1}{2} \int \overbrace{f(\tau) f(t-\tau)}^{0} d\tau = 0$.

Also notice that $g(t+2) = \frac{1}{2} \int_{-1}^{1} f(\tau) f(t+2-\tau)\, d\tau$

since $f$ is periodic with $T=2$.

$= \frac{1}{2} \int_{-1}^{1} f(\tau) f(t-\tau)\, d\tau = g(t) \Rightarrow g(t)$ is periodic with $T=2$

$\Rightarrow g(t)$



(4.) (a) Since $y(t) = H(j\omega)e^{j\omega t}$, $y'(t) = j\omega \cdot H(j\omega)e^{j\omega t} = j\omega \cdot H(j\omega)x(t)$.

Also, $x(t-1) = e^{j\omega(t-1)} = e^{-j\omega}x(t)$.

Therefore, $y(t) = \left(\frac{3}{4}j\omega \cdot H(j\omega) + 1 - e^{-j\omega}\right) \cdot x(t) = H(j\omega) \cdot x(t)$

$$\Rightarrow H(j\omega) = \frac{3}{4}j\omega \cdot H(j\omega) + 1 - e^{-j\omega}$$

$$\Rightarrow H(j\omega) = \frac{1 - e^{-j\omega}}{1 - \frac{3}{4}j\omega}$$

(b) Since for $z(t) = e^{-st}u(t)$, (with $s>0$)

$$Z(j\omega) = \int_0^\infty e^{-st}e^{-j\omega t}\,dt = \frac{1}{s+j\omega}$$

we have that the FT of $z(-t)$ is given by

$$Z(-j\omega) = \frac{1}{s-j\omega}$$

Therefore the inverse FT of $\frac{1}{1 - 3/4\, j\omega} = \frac{1}{3/4\,(4/3 - j\omega)} = H_1(j\omega)$

is $\frac{4}{3}e^{4/3t}u(-t)$. $(= h_1(t))$

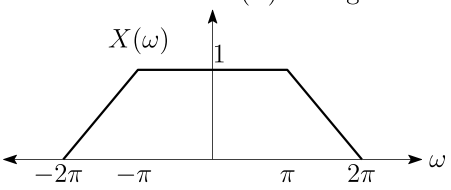
Finally the FT of $h_1(t-1)$ is $e^{-j\omega} \cdot H_1(j\omega)$

so, $h(t) = h_1(t) - h_1(t-1) = \frac{4}{3}\left[e^{4/3t}u(-t) - e^{4/3(t-1)} \cdot u(1-t)\right]$

# EHB 252E – Homework 4

Due 10.05.2012

1. Let $x(t)$ be a continuous-time signal whose Fourier transform $X(\omega)$ is as given below.

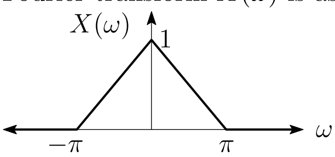



   (a) Let $f(n)$ be a discrete time signal defined as $f(n) = x(n/2)$. Determine and sketch $F(e^{j\omega})$.

   (b) Let $f(n)$ be a discrete time signal defined as $f(n) = x(n)$. Determine and sketch $F(e^{j\omega})$.

   (c) Let $f(n)$ be a discrete time signal defined as $f(n) = x(2\,n)$. Determine and sketch $F(e^{j\omega})$.

2. Let $x(t)$ be a continuous-time signal whose Fourier transform $X(\omega)$ is as given below.




   Also let

$$p(t) = \sum_{k\in\mathbb{Z}} \delta(t - 2\,k) \quad \text{and} \quad s(t) = p(t-1).$$

   Using these, we define two new signals $g(t)$, $h(t)$ as, $g(t) = x(t)\,p(t)$ and $h(t) = x(t)\,s(t)$.

   (a) Sketch $p(t)$.

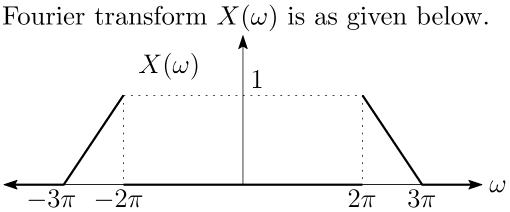
   (b) Determine and sketch $G(\omega)$, the Fourier transform of $g(t)$.

   (c) Sketch $s(t)$.

   (d) Determine and sketch $H(\omega)$, the Fourier transform of $h(t)$.

   (e) Let $y(t) = g(t) + h(t)$. Determine and sketch $Y(\omega)$, the Fourier transform of $y(t)$.

3. Let $x(t)$ be a continuous-time signal whose Fourier transform $X(\omega)$ is as given below.



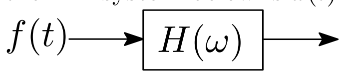

   Also, let

$$f(t) = \sum_{k\in\mathbb{Z}} x(k)\,\delta(t-k).$$

   (a) What is the Nyquist frequency for $x(t)$?

   (b) Sketch $F(\omega)$, the Fourier transform of $f(t)$.

   (c) Find a filter $H(\omega)$ so that the output of the LTI system below is $x(t)$.




   (d) Find a function $g(t)$ so that

$$x(t) = \sum_{k\in\mathbb{Z}} x(k)\,g(t-k).$$

4. Suppose that $x(n)$ is a discrete-time sequence whose DTFT is specified as,

$$X(e^{j\omega}) = \begin{cases} 1, & \text{for } |\omega| \le \pi/2, \\ 0, & \text{for } \pi/2 < |\omega| \le \pi. \end{cases}$$

   (a) Sketch $X(e^{j\omega})$ for $-2\pi \le \omega \le 2\pi$.

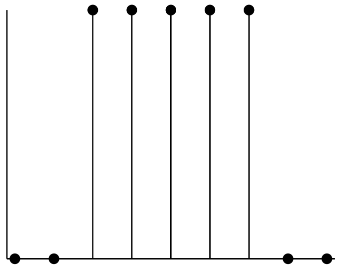
   (b) Let $y(n)$ be defined as,

$$y(n) = \begin{cases} x(n/3), & \text{if } n \text{ is divisible by 3,} \\ 0, & \text{if } n \text{ is not divisible by 3.} \end{cases}$$

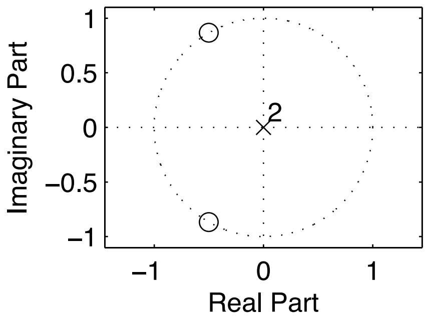
   Sketch $Y(e^{j\omega})$ for $-\pi \le \omega \le \pi$.

   (c) Let $z(n)$ be defined as $z(n) = x(3n)$. Sketch $Z(e^{j\omega})$ for $-2\pi \le \omega \le 2\pi$.

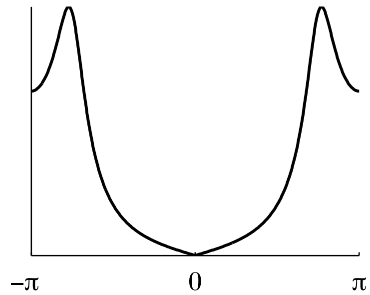
5. Below are shown four discrete-time signals, their DTFT magnitudes, and pole-zero diagrams. But they are not in the correct order. Put them in the correct order by matching each signal with its DTFT magnitude and pole-zero diagram.



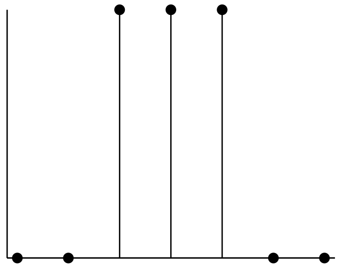
(I)



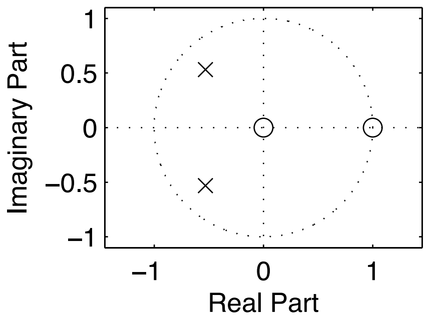

(a)



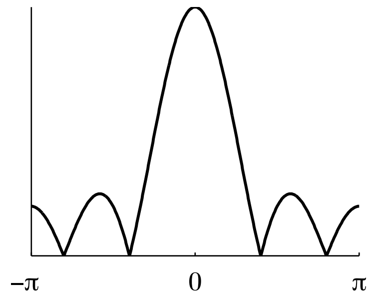

(1)



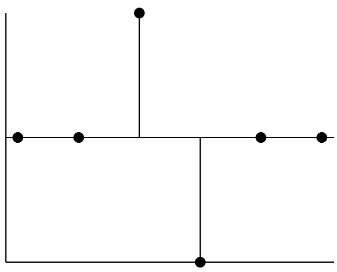
(II)



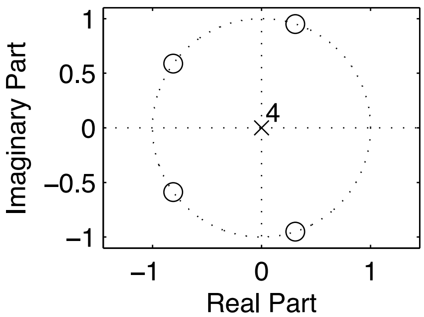

(b)



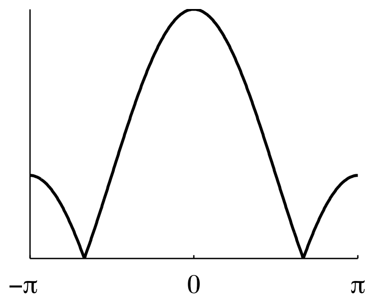

(2)



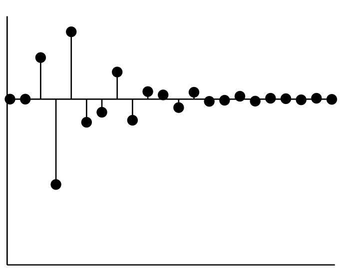
(III)



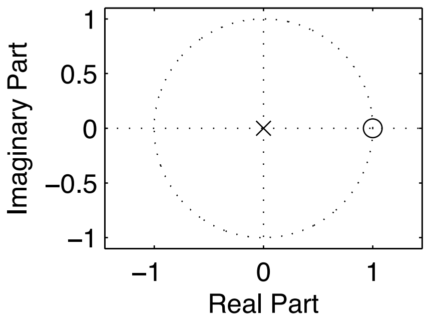

(c)



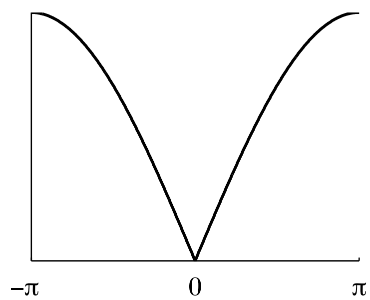

(3)



(IV)




(d)




(4)

EHB 252E – Signals and Systems

Midterm Examination I

15.03.2012

4 Questions, 100 Minutes

Please Show Your Work!

(25 pts) 1. Let $f(t)$ be a continuous-time signal described as

$$f(t) = \begin{cases} 1+t, & \text{for } -1 \le t \le 0, \\ 1, & \text{for } 0 < t \le 1, \\ 0, & \text{for } t \notin [-1,1]. \end{cases}$$

(a) Sketch $f(t)$.

(b) Let $g(t) = f\left(\frac{1}{2}\,t\right)$. Determine and sketch $g(t)$.

(c) Let $h(t) = f(2-t)$. Determine and sketch $h(t)$.

(25 pts) 2. Consider the discrete-time LTI system below.

$$x(n) \longrightarrow \boxed{S} \longrightarrow y(n)$$

- If we input $x_1(n) = \delta(n-1) + \delta(n-2)$ to the system, the output is $y_1(n) = \delta(n) - \delta(n-2)$.
- If we input $x_2(n) = 2\,\delta(n) + \delta(n-1)$ to the system, the output is $y_2(n) = 2\,\delta(n+1) - \delta(n) - \delta(n-1)$.

(a) Determine and sketch the output if we input $x_1(n-1)$ to the system.

(b) Determine and sketch the output if we input $x_1(n) - x_2(n)$ to the system.

(c) Determine and sketch the impulse response of $S$.

(25 pts) 3. Consider the discrete-time, <u>time-invariant</u> system below.

$$x(n) \longrightarrow \boxed{S} \longrightarrow y(n)$$

Suppose that the relation between $y(0)$ and the input is,

$$y(0) = 2\left[x(0)\right]^2 + \left[x(1)\right]^2$$

(a) Determine the output if we input $x_1(n) = 2\,\delta(n)$ to the system.

(b) Determine the output if we input $x_2(n) = -\delta(n-1)$ to the system.

(c) Determine the output if we input $x_3(n) = 2\,\delta(n) - \delta(n-1)$ to the system.

(25 pts) 4. Let $F_1$ and $F_2$ be continuous-time LTI systems. We connect these two systems in cascade to form the system $S$ shown below.

$$x(t) \longrightarrow \boxed{F_1} \longrightarrow \boxed{F_2} \longrightarrow y(t)$$

Suppose that

- The impulse response of $S$ is

$$h(t) = \begin{cases} t, & \text{for } 0 \leq t \leq 1, \\ 2 - t, & \text{for } 1 < t \leq 2, \\ 0, & \text{for } t \notin [0, 2]. \end{cases}$$

- The impulse response of $F_2$ is

$$u(t) = \begin{cases} 1, & \text{for } 0 \leq t, \\ 0 & \text{for } t < 0. \end{cases}$$

(a) Determine the step response of $S$ (i.e. the response of the system when a unit step function is input).

(b) Determine the step response of $F_1$.

(c) Determine the impulse response of $F_1$.

EHB 252E – Signals and Systems

Midterm Examination II

19.04.2012

Student Name : ____________________

Student Num. : ____________________

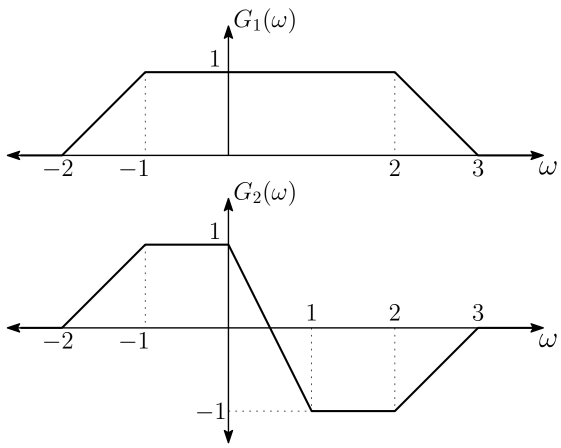
4 Questions, 90 Minutes

Please Show Your Work!

(25 pts) 1. Let $g(t)$ be a continuous time-signal. Suppose that the Fourier transforms of

$$g_1(t) = g(t)\cos(t),$$
$$g_2(t) = -j\,g(t)\sin(t),$$

are as shown below (both $G_1(\omega)$ and $G_2(\omega)$ turn out to be real-valued functions).




Determine and sketch $G(\omega)$, the Fourier transform of $g(t)$.

(25 pts) 2. Let $S_1$ and $S_2$ be causal, LTI, continuous-time systems as shown below.

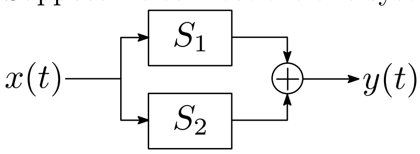
$$x(t) \longrightarrow \boxed{S_1} \longrightarrow y_1(t) \qquad\qquad x(t) \longrightarrow \boxed{S_2} \longrightarrow y_2(t)$$

Suppose that the input and the ouput of $S_1$ and $S_2$ satisfy the differential equations :

$$S_1 : \quad y_1(t) + \frac{1}{2}y_1'(t) = x(t)$$
$$S_2 : \quad y_2(t) + \frac{1}{4}y_2'(t) = x(t)$$

(a) Find $h_1(t)$, the impulse response of $S_1$.

(b) Suppose we connect the two systems in parallel to form the system $S$ as shown below.




For $S$, find the differential equation that relates $y(t)$ to $x(t)$.

(25 pts) 3. Consider a discrete-time LTI system whose unit impulse response is $h(n) = \alpha^{n-n_0}\, u(n-n_0)$ (where $|\alpha| < 1$ and $n_0$ is a constant). Suppose we input $x(n) = e^{j\pi n}$ to the system. Also, let $y(n)$ denote the output.

(a) Find $H(e^{j\omega})$, the frequency response of the system (i.e. the DTFT of the impulse response), in terms of $\alpha$ and $n_0$.

(b) Determine $X(e^{j\omega})$, the DTFT of the input.

(c) Determine $Y(e^{j\omega})$, the DTFT of the output, in terms of $\alpha$ and $n_0$.

(d) Determine the output $y(n)$, for $\alpha = 2/3$, $n_0 = 4$.

(25 pts) 4. Consider a causal, discrete-time LTI system with impulse response $h(n)$. Suppose that the input $x(n)$ and the output $y(n)$ satisfy the difference equation

$$y(n) - \frac{1}{4}\, y(n-1) = x(n-2).$$

(a) Find the frequency response $H(e^{j\omega})$ of the system (i.e. the DTFT of $h(n)$).

(b) Find the impulse response $h(n)$ of the system.

(c) Find the output if we input $x(n) = (1/2)^n\, u(n)$ to the system.

EHB 252E – Signals and Systems

Final Examination

22.05.2012

5 Questions, 120 Minutes

Please Show Your Work!

(25 pts) 1. Below, the relation between the input, $x(t)$, and the output, $y(t)$, are given for five different continuous-time systems. For each system, specify whether the system is (i) linear or not, (ii) time-invariant or not. Briefly explain your answer for full credit.

(a) $y(t) = 2x(t)$.

(b) $y(t) = 2x(t) + 1$.

(c) $y(t) = x(t^2 + 1)$.

(d) $y(t) = t\,x(t^2 + 1)$.

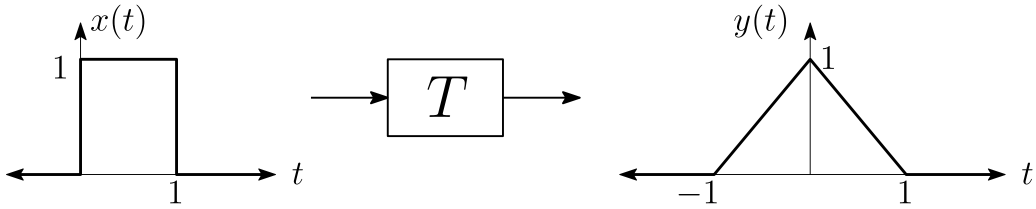
(e) $y(t) = \int_0^2 \tau\,x(t - \tau)\,d\tau$.

(15 pts) 2. Consider an LTI continuous-time system where the relation between the input $x(t)$ and the output $y(t)$ is specified as,

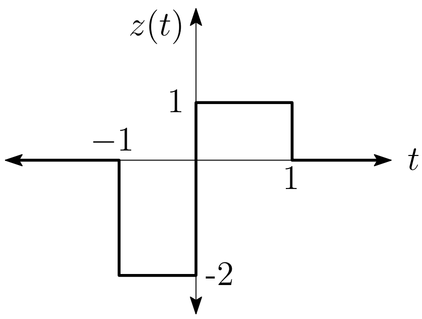
$$y(t) = \int_{t-1}^{t+1} x(\tau)\,d\tau.$$

Determine and sketch the impulse response of this system.

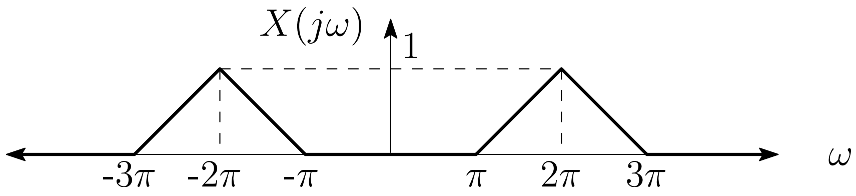
(15 pts) 3. Let $T$ be a continuous-time LTI system. Suppose that, if we input $x(t)$ to the system, the output is $y(t)$ as shown below.




Determine and sketch the output if we input $z(t)$ shown below to $T$.



(20 pts) 4. Let $x(t)$ be a continuous-time signal whose Fourier transform is as shown below.




Also, let the continuous-time signal $y(t)$ be defined as,

$$y(t) = \sum_{k \in \mathbb{Z}} x(k/2)\, \delta(t - k/2).$$

Finally, let us define the discrete-time signal $z(n)$ as,

$$z(n) = x(n/2) \quad \text{for } n \in \mathbb{Z}.$$

(a) Determine and sketch $Y(j\omega)$, the Fourier Transform of $y(t)$.

(b) Determine and sketch $Z(e^{j\omega})$, the DTFT of $z(n)$.

(25 pts) 5. Let $T$ be an LTI system with impulse response $h(n) = (1/2)^n\, u(n)$. Also, let the step response of this system be $y(n)$.

(a) Find $H(z)$, the $z$-transform of the impulse response.

(b) Sketch the pole-zero diagram of $H(z)$ and specify the region of convergence (ROC).

(c) Find $Y(z)$, the $z$-transform of the step response. What is the region of convergence?

(d) Determine $y(n)$, the step response of the system.