EHB 252E – Signals and Systems

Spring 2013

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Class Meets :	13.30 – 16.30, Thursday EEB 5204
Office Hours :	10.00 – 12.00, Thursday
Textbook :	A. V. Oppenheim, A. S. Willsky, S. H. Nawab, 'Signals and Systems', 2^{nd} Edition, Prentice Hall.
Grading :	2 Midterms (30% each), Final (40%).

Tentative Course Outline

- Introduction to Signals and Systems

  Basic discrete and continuous-time signals and their properties, transformations of the independent variable, basic system properties

- Linear Time-Invariant Systems

  Discrete and continuous time LTI systems, convolution sum and integral, properties of LTI systems, LTI systems defined by differential and difference equations

- Fourier Series

  Representation of continuous and discrete-time signals using Fourier series, properties of Fourier series, LTI systems and Fourier series

- The Continuous-Time Fourier Transform

- The Discrete-Time Fourier Transform

- Frequency Domain Characterization of Systems

  Magnitude-phase representations, ideal frequency selective filters

- Sampling

  A/D, D/A conversion, aliasing, discrete-time processing of continuous-time signals

- The z-Transform

  The z transform, region of convergence, inverse z-transform, properties of the z-transform, characterizing LTI systems using the z-transform

- The Laplace Transform

  The Laplace transform, region of convergence, inverse Laplace transform, properties of the Laplace transform, characterizing LTI systems using the Laplace transform

EHB 252E – Homework 1

Due 28.02.2013

1. Let $x(t)$ be a signal described as

$$x(t) = \begin{cases} t+1, & \text{for } -1 \leq t \leq 0 \\ 1, & \text{for } 0 < t \leq 1 \\ 0, & \text{for } t \notin [-1,1]. \end{cases}$$

   (a) Sketch $x(t)$.

   (b) Sketch $y(t) = x(2-t)$.

2. Consider a continuous-time system T as shown below.

$$x(t) \longrightarrow \boxed{T} \longrightarrow y(t)$$

   (a) Suppose that $y(t) = x(2t)$. Is T causal? Linear? Time-invariant? Stable in the BIBO sense?

   (b) Suppose that $y(t) = \int_{-1}^{1} x(t-s)\, s^2\, ds$. Is T linear? Time-invariant? Stable in the BIBO sense?

   (c) Suppose that $y(t) = x'(t)$ (i.e. the system differentiates the input). Is T linear? Time-invariant? Stable in the BIBO sense?

Solution. In the following, suppose that the output to the input $x_i(t)$ is denoted by $T\{x_i\} = y_i(t)$ for $i = 1, 2$.

(a) Note that $y(1) = x(2)$, so T is not causal. For a, b real numbers, we have that

$$T\{\underbrace{a\,x_1(t) + b\,x_2(t)}_{z(t)}\} = z(2t) = a\,x_1(2t) + b\,x_2(2t) = a\,y_1(t) + b\,y_2(t).$$

Therefore, T is linear.

Suppose $x_2(t) = x_1(t-1)$. Then, $y_2(t) = x_2(2t) = x_1(2\,t-2) \neq y_1(t-1)$. Thus, T is not time-invariant.

Suppose $|x(t)| < M$ for some M. Then, we also have that $|y(t)| < M$. Therefore the system is BIBO stable.

(b) For the inputs x_1, x_2, we have,

$$\begin{aligned} y_1(t) &= \int_{-1}^{1} x_1(t-s)\, s^2\, ds, \\ y_2(t) &= \int_{-1}^{1} x_2(t-s)\, s^2\, ds. \end{aligned}$$

Therefore if we input $a\,x_1(t) + b\,x_2(t)$ to the system, we get

$$\begin{aligned} y(t) &= \int_{-1}^{1} (a\,x_1(t-s) + b\,x_2(t-s))\, s^2\, ds \\ &= a \int_{-1}^{1} x_1(t-s) ds + b \int_{-1}^{1} x_2(t-s))\, s^2\, ds \\ &= a\,y_1(t) + b\,y_2(t). \end{aligned}$$

Therefore, the system is linear.

Suppose $x_2(t) = x_1(t-\tau)$ for some τ. Then,

$$\begin{aligned} y_2(t) &= \int_{-1}^{1} x_2(t-\tau)\, s^2\, ds \\ &= \int_{-1}^{1} x_1(t-\tau-s)\, s^2\, ds \\ &= y_1(t-\tau). \end{aligned}$$

Since τ is arbitrary, T is time-invariant.
Suppose now that $|x(t)| < M$ for all t. We have then,

$$\begin{aligned} |y(t)| &= \left| \int_{-1}^{1} x(t-\tau)\, s^2\, ds \right| \\ &\leq \int_{-1}^{1} \left| x_1(t-\tau-s)\, s^2 \right| ds \\ &\leq \int_{-1}^{1} M \left| s^2 \right| ds \\ &= \frac{2}{3} M. \end{aligned}$$

Thus, the system is BIBO-stable.

(c) Since $\big(a\, x_1(t) + b\, x_2(t)\big)' = a\, x_1'(t) + b\, x_2'(t) = a\, y_1(t) + b\, y_2(t)$, the system is linear.
Also, noting that $\big(x_1(t-\tau)\big)' = x_1'(t-\tau) = y_1(t-\tau)$, the system is seen to be time-invariant.
Finally, consider the family of signals

$$x_\epsilon(t) = \begin{cases} t/\epsilon, & \text{for } 0 \leq t \leq \epsilon, \\ 0, & \text{for } t \notin [0, \epsilon]. \end{cases}$$

Observe that $|x_\epsilon(t)| \leq 1$. However, we have that $|x_\epsilon(\epsilon/2)| = 1/\epsilon^2$. Even though this family of functions are bounded with the same bound, namely 1, the system output for this family is not bounded. Therefore the system is not BIBO-stable.

3. An AM modulator transforms a given signal $x(t)$ into $y(t) = \cos\big(2\pi\, f\, t\big)\, x(t)$, where f is a fixed constant (called the modulating frequency). Is the AM modulator linear? Time-invariant?

Solution. Suppose that the output to the input $x_i(t)$ is denoted by $T\{x_i\} = y_i(t)$ for $i = 1, 2$. We have then,

$$\begin{aligned} T\{a\, x_1(t) + b\, x_2(t)\} &= \cos\big(2\pi\, f\, t\big)\, \big(a\, x_1(t) + b\, x_2(t)\big) \\ &= a\, \big[\cos\big(2\pi\, f\, t\big)\, x_1(t)\big] + b\, \big[\cos\big(2\pi\, f\, t\big)\, x_2(t)\big] \\ &= a\, y_1(t) + b\, y_2(t). \end{aligned}$$

Therefore the system is linear.
Now suppose that $x_2(t) = x_1(t - 1/(2f))$. In this case,

$$\begin{aligned} T\{x_2(t)\} &= \cos\big(2\pi\, f\, t\big)\, x_1(t - 1/(2f)) \\ &= \cos\big(2\pi\, f\, (t - 1/(2f) + 1/(2f))\big)\, x_1(t - 1/(2f)) \\ &= -\cos\big(2\pi\, f\, (t - 1/(2f))\big)\, x_1(t - 1/(2f)) \\ &= -y_1(t - 1/(2f)) \neq y_1(t - 1/(2f)). \end{aligned}$$

Thus, the system is not time-invariant.

EHB 252E – Homework 2

Due 07.03.2013

1. Let S_1 and S_2 be two systems, connected in cascade as shown below.

   $$\longrightarrow \boxed{S_1} \longrightarrow \boxed{S_2} \longrightarrow$$

   Let us call the overall system S.

   (a) Suppose that S_1 and S_2 are both linear systems. Show that S is also a linear system.

   (b) Suppose that S_1 and S_2 are both time-invariant systems. Show that S is also a time-invariant system.

   Solution. (a) Suppose we input x_1 and x_2 to the cascade system and observe at the output y_1 and y_2 respectively. Now suppose also that the output of S_1 to the input x_i is given by z_i for $i = 1, 2$. In this case, we should have that if we input z_i to S_2 then we should observe at the output y_i, for $i = 1, 2$.

   Now suppose we input $x = a\,x_1 + b\,x_2$ to the system S_1, where a and b are real constants. Because of the linearity of S_1, the output should be $z = a\,z_1 + b\,z_2$. Now if we input z to S_2, since (i) z is a linear combination of z_i's, (ii) the response of S_2 to z_i is y_i and (iii) S_2 is a linear system, we should observe at the output $y = a\,y_1 + b\,y_2$. Thus, the response of the cascade system is also the same linear combination of y_i's. Therefore the cascade system is linear.

   (b) Suppose that when we input $x_1(t)$ to S_1, the output is $z_1(t)$, and when $z_1(t)$ is input to S_2, the output is $y_1(t)$. This means that when $x_1(t)$ is input to the system connected in series, the output is $y_1(t)$.

   Now suppose we input $x_2(t) = x_1(t - \tau)$ to S_1, where τ is a constant. Because of the time-invariance of S_1, the output should satisfy $z_2(t) = z_1(t - \tau)$. Now if we input $z_2(t)$ to S_2, by the time-invariance of S_2, the output should satisfy $y_2(t) = y_1(t - \tau)$. In summary, we found that, provided that both systems are time-invariant, when we shift the input to the system in cascade, the output is shifted by the same amount. Therefore the system connected in cascade is also time-invariant.

2. (From our textbook) Given a signal $f(t)$, let us define c_f as,

   $$c_f = \int_{-\infty}^{\infty} f(t)\,dt.$$

   Now let $h(t)$ be the impulse response of an LTI system. Also, let $y(t)$ be the output of this system when $f(t)$ is input, i.e. $y(t) = (h * f)(t)$. Show that

   $$c_y = c_h\,c_f.$$

   Solution. Note that

   $$y(t) = \int_{-\infty}^{\infty} h(t - \tau)\,f(\tau)\,d\tau.$$

   Therefore, (assuming we can change the order of integration...)

   $$\begin{aligned}
   c_y &= \int_{-\infty}^{\infty} y(t)\,dt \\
   &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(t-\tau)\,f(\tau)\,d\tau \right) dt \\
   &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(t-\tau)\,dt \right) f(\tau)\,d\tau \\
   &= c_h \int_{-\infty}^{\infty} f(\tau)\,d\tau \\
   &= c_h\,c_f.
   \end{aligned}$$

3. Consider a discrete-time LTI system whose impulse response is $h(n) = \delta(n) - \delta(n-1)$. We input $x(n)$ to this system and observe the output $y(n) = \delta(n+1) - \delta(n-2)$. Suppose we also know that $x(-3) = 0$. Determine $x(n)$.

   Solution. Observe that

   $$(x * h)(n) = x(n) - x(n-1).$$

   Therefore, we should have

   $$y(n) = x(n) - x(n-1).$$

   But we are already given what $y(n)$ is. So we obtain that (i) $x(n) - x(n-1) = \delta(n+1) - \delta(n-2)$, (ii) $x(-3) = 0$. Writing (i) as $x(n) = x(n-1) + \delta(n+1) - \delta(n-2)$, we can make use of (ii) to compute $x(-2)$ as 0. Feeding this into the same equation, we obtain $x(-1)$ as 1. Continuing like this, we find

   $$\begin{aligned}
   x(0) &= x(-1) + \delta(1) - \delta(-2) = 1, \\
   x(1) &= x(0) + \delta(2) - \delta(-1) = 1, \\
   x(2) &= x(1) + \delta(3) - \delta(0) = 0, \\
   x(3) &= x(2) + \delta(4) - \delta(1) = 0,
   \end{aligned}$$

   Observe also that $x(n) = 0$, for $n > 3$. Similarly, by rewriting (i) above as $x(n-1) = x(n) - \delta(n+1) + \delta(n-2)$ we can see that $x(n) = 0$ for $n < -3$.

   Thus, $x(n) = \delta(n+1) + \delta(n) + \delta(n-1)$.

4. (a) Consider an LTI system whose output $y(t)$ is related to its input $x(t)$ through the equation

   $$y(t) = 2 \int_0^3 x(t-\tau)\, \tau \, d\tau.$$

   Find the impulse response of this system.

   (b) Consider an LTI system whose output $y(t)$ is related to its input $x(t)$ through the equation

   $$y(t) = \int_t^{t+1} x(\tau - 2)\, (1 - t + \tau)\, d\tau.$$

   Find the impulse response of this system.

   Solution. (a) Notice that $y(t)$ can be written as

   $$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau,$$

   where

   $$h(t) = \begin{cases} 2, & \text{for } t \in [0,3], \\ 0, & \text{for } t \notin [0,3]. \end{cases}$$

   This $h(t)$ should be the impulse response of the system.

   (b) Let us change variables and define $s = \tau - 2$. Then we have, (observe also the change of the limits of integration)

   $$\begin{aligned}
   y(t) &= \int_{t-2}^{t-1} x(s)\, (1 - t + s + 2)\, ds \\
   &= \int_{-\infty}^{\infty} x(s)\, \big(3 - (t-s)\big) \Big(u\big(s - (t-2)\big) - u\big(s - (t-1)\big) \Big)\, ds \\
   &= \int_{-\infty}^{\infty} x(s)\, \big(3 - (t-s)\big) \Big(u\big(2 - (t-s)\big) - u\big(1 - (t-s)\big) \Big)\, ds \\
   &= \int_{-\infty}^{\infty} x(s)\, h(t-s)\, ds,
   \end{aligned}$$

where

$$h(t) = \begin{cases} 3 - t, & \text{for } t \in [1, 2], \\ 0, & \text{for } t \notin [1, 2]. \end{cases}$$

This $h(t)$ should be the impulse response of the system.

EHB 252E – Homework 3

Due 21.03.2013

1. Let S be a discrete-time LTI system. Suppose we input

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

to S and observe the output

$$y(n) = \delta(n) + \left(\frac{1}{2}\right)^{n-1} u(n-1).$$

Find $h(n)$, the impulse response of S.

2. Let $x(t) = e^{2\,t}\,u(-t)$, $h(t) = u(t)$. Also, let $y(t) = (x * h)(t)$. Determine and sketch $y(t)$.

3. Consider an LTI system whose response to the input $x(t) = \delta(t) - \delta(t-1)$ is given by

$$h(t) = \begin{cases} 1, & \text{for } 0 < t \leq 1, \\ -1, & \text{for } 1 < t \leq 2. \end{cases}$$

Determine the unit step response of this system.

4. Suppose that, for a given signal $x(t)$ (not necessarily periodic), we define

$$y(t) = \sum_{k=-\infty}^{\infty} x(t - kT).$$

Assume that $|y(t)| \leq \infty$.

(a) Show that $y(t)$ is periodic by T.

(b) Find a function $f(t)$ such that

$$\int_0^T y(t)\,\phi_k^*(t)\,dt = \int_{-\infty}^{\infty} x(t)\,f(t)\,dt,$$

where

$$\phi_k(t) = \exp\left(j\frac{2\pi}{T}k\,t\right).$$

EHB 252E – Homework 3 Solutions

1. Notice that

$$\left(\frac{1}{2}\right)^n u(n) = \delta(n) + \left(\frac{1}{2}\right)^n u(n-1).$$

Using this observation, we can write

$$\begin{aligned}
y(n) &= y(n) + \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n) \\
&= \left(\frac{1}{2}\right)^n u(n) + \left[\left(\frac{1}{2}\right)^{n-1} u(n-1) - \left(\frac{1}{2}\right)^n u(n-1)\right] + \left[\delta(n) - \delta(n)\right] \\
&= \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^{n-1} u(n-1) \\
&= x(n) - x(n-1).
\end{aligned}$$

Thus, $h(n) = \delta(n) - \delta(n-1)$.

2. We have,

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} e^{2\tau}\, u(-\tau)\, u(t-\tau)\, d\tau.
\end{aligned}$$

We will consider two special intervals for t.

(i) $t < 0$: In this case,

$$u(-\tau)\, u(t-\tau) = \begin{cases} 1, & \text{for } \tau \leq t, \\ 0, & \text{for } t < \tau. \end{cases}$$

Therefore, for $t < 0$,

$$\begin{aligned}
y(t) &= \int_{-\infty}^{t} e^{2\tau}\, d\tau \\
&= \frac{1}{2}\, e^{2t}.
\end{aligned}$$

(ii) $t \geq 0$: In this case,

$$u(-\tau)\, u(t-\tau) = \begin{cases} 1, & \text{for } \tau \leq 0, \\ 0, & \text{for } 0 < \tau. \end{cases}$$

Therefore, for $t \geq 0$,

$$\begin{aligned}
y(t) &= \int_{-\infty}^{0} e^{2\tau}\, d\tau \\
&= \frac{1}{2}.
\end{aligned}$$

To summarize,

$$y(t) = \begin{cases} \frac{1}{2}\, e^{2t}, & \text{for } t < 0, \\ \frac{1}{2}, & \text{for } 0 \leq t. \end{cases}$$

3. Observe that if we convolve $x(t)$ and $h(t)$, we obtain (carry out this convolution!),

$$z(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq 1, \\ 0, & \text{for } t \notin [0,1]. \end{cases}$$

Also observe that

$$u(t) = \sum_{k=0}^{\infty} z(t-k).$$

Therefore if the response of the system to $z(t)$ is given as $y(t)$, then by the LTI property, the unit step response should be

$$s(t) = \sum_{k=0}^{\infty} y(t-k).$$

To find the response of the system to $z(t)$, namely $y(t)$, all we have to do is to convolve $h(t)$ with $u(t)$ (why?). Convolving $h(t)$ and $u(t)$ (make sure you can do this on your own!), we obtain (sketch it to see what it looks like),

$$y(t) = \begin{cases} t, & \text{for } 0 \leq t \leq 1, \\ 2-t, & \text{for } 1 < t \leq 2, \\ 0, & \text{for } t \notin [0,2]. \end{cases}$$

From $y(t)$, we obtain $s(t)$ as (sketch $y(t)$, $y(t-1)$, $y(t-2)$, etc. to see this)

$$s(t) = \sum_{k=0}^{\infty} y(t-k) \begin{cases} 0, & \text{for } t < 0, \\ t, & \text{for } 0 \leq t \leq 1, \\ 1, & \text{for } 1 < t. \end{cases}$$

4. (a) We claim that $y(t)$ is periodic with T. To show this, observe that

$$\begin{aligned} y(t-T) &= \sum_{k=-\infty}^{\infty} x\big((t-T)-kT\big) \\ &= \sum_{k=-\infty}^{\infty} x\big(t-(k+1)T\big) \\ &= \sum_{l=-\infty}^{\infty} x\big(t-lT\big) \quad \Big(\text{take } l = k+1\Big) \\ &= y(t). \end{aligned}$$

(b) Observe that $\phi_k^*(t+T) = \phi_k^*(t)$. That is, $\phi_k^*(t)$ is periodic with T. Now,

$$\begin{aligned} \int_0^T y(t)\,\phi_k^*(t)\,dt &= \int_0^T \sum_{k=-\infty}^{\infty} x(t-kT)\,\phi_k^*(t)\,dt \\ &= \sum_{k=-\infty}^{\infty} \int_0^T x(t-kT)\,\phi_k^*(t)\,dt \\ &= \sum_{k=-\infty}^{\infty} \int_0^T x(t-kT)\,\phi_k^*(t-kT)\,dt \\ &= \sum_{k=-\infty}^{\infty} \int_{kT}^{(k+1)\,T} x(t)\,\phi_k^*(t)\,dt \\ &= \int_{-\infty}^{\infty} x(t)\,\phi_k^*(t)\,dt \end{aligned}$$

Therefore, $\phi_k^*(t)$ is the function $f(t)$ we are looking for.

EHB 252E – Homework 4

Due 25.04.2013

1. Suppose that for an LTI system, if we input $x(n) = 3^{-n}\,u(n)$, the output is $y(n) = 4^{-n}\,u(n)$.

   (a) Find a difference equation that relates the input and the output of this system.

   (b) Find the impulse response of this system.

   Solution. (a) Notice that

   $$\delta(n) = x(n) - \frac{1}{3}\,x(n-1).$$

   Similarly, we can write

   $$\delta(n) = y(n) - \frac{1}{4}\,y(n-1).$$

   Equating the left hand sides of these equations, we obtain,

   $$y(n) - \frac{1}{4}\,y(n-1) = x(n) - \frac{1}{3}\,x(n-1).$$

   (b) Taking the DTFT of the difference equation above, we obtain the frequency response of the system as,

   $$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1 - \frac{1}{3}\,e^{-j\omega}}{1 - \frac{1}{4}\,e^{-j\omega}} \\ &= 1 - \frac{\frac{1}{12}\,e^{-j\omega}}{1 - \frac{1}{4}\,e^{-j\omega}}. \end{aligned}$$

   Thus, the impulse response is

   $$h(n) = \delta(n) - \frac{1}{12}\,4^{-(n-1)}\,u(n-1).$$

2. Suppose that the input $x(n)$ and the output $y(n)$ of a causal LTI system satisfy the difference equation

   $$y(n) - \frac{1}{4}\,y(n-1) = x(n) - x(n-1).$$

   Find the output if we input $x(n) = (1/2)^n\,u(n)$ to the system.

   Solution. Taking the DTFT of both sides of the difference equation, we obtain,

   $$Y(e^{j\omega})\left(1 - \frac{1}{4}\,e^{-j\omega}\right) = X(e^{j\omega})\left(1 - e^{-j\omega}\right)$$

   Now, for $x(n) = (1/2)^n\,u(n)$, we have,

   $$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}\,e^{-j\omega}}.$$

Therefore,

$$Y(e^{j\omega}) = \frac{1 - e^{-j\omega}}{\left(1 - \frac{1}{2}\, e^{-j\omega}\right)\left(1 - \frac{1}{4}\, e^{-j\omega}\right)}$$
$$= \frac{3}{1 - \frac{1}{4}\, e^{-j\omega}} - \frac{2}{1 - \frac{1}{2}\, e^{-j\omega}}.$$

We recognize the inverse DTFT of this as,

$$y(n) = 3\left(\frac{1}{4}\right)^n u(n) - 2\left(\frac{1}{2}\right)^n u(n)$$

3. (Derived from our book) We are given a discrete-time LTI system whose input $x(n)$ and output $y(n)$ are related by the pair of equations

$$y(n) + \frac{1}{4}\, y(n-1) + z(n) + \frac{1}{2}\, z(n-1) = \frac{2}{3}\, x(n)$$
$$y(n) - \frac{5}{4}\, y(n-1) + z(n) - z(n-2) = x(n),$$

where $z(n)$ denotes an intermediate signal. Find a single difference equation that relates $y(n)$ to $x(n)$.

Solution. Taking the DTFT of the first equation, we obtain,

$$Y(e^{j\omega})\left(1 + \frac{1}{4}\, e^{-j\omega}\right) + Z(e^{j\omega})\left(1 + \frac{1}{2}\, e^{-j\omega}\right) = \frac{2}{3}\, X(e^{j\omega}).$$

Rearranging, we have,

$$Z(e^{j\omega}) = \frac{1}{1 + \frac{1}{2}\, e^{-j\omega}}\left[\frac{2}{3}\, X(e^{j\omega}) - Y(e^{j\omega})\left(1 + \frac{1}{4}\, e^{-j\omega}\right)\right].$$

Similarly, taking the DTFT of the second given equation and rearranging, we have,

$$Z(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}}\left[X(e^{j\omega}) - Y(e^{j\omega})\left(1 - \frac{5}{4}\, e^{-j\omega}\right)\right].$$

Now equating the last two equations, we obtain,

$$(1 - e^{-j\omega})\left[\frac{2}{3}\, X(e^{j\omega}) - Y(e^{j\omega})\left(1 + \frac{1}{4}\, e^{-j\omega}\right)\right]$$
$$= \left(1 + \frac{1}{2}\, e^{-j\omega}\right)\left[X(e^{j\omega}) - Y(e^{j\omega})\left(1 - \frac{5}{4}\, e^{-j\omega}\right)\right].$$

Taking the inverse DTFT, we obtain,

$$\frac{2}{3}\left(x(n) - x(n-1)\right) - \left(y(n) - \frac{3}{4}\, y(n-1) - \frac{1}{4}\, y(n-2)\right)$$
$$= x(n) + \frac{1}{2}\, x(n-1) - \left(y(n) - \frac{3}{4}\, y(n-1) - \frac{5}{8}\, y(n-2)\right).$$

Rearranging,

$$\frac{3}{8}\, y(n-2) = -\frac{1}{3}\, x(n) - \frac{7}{6}\, x(n-1).$$

Or,

$$y(n) = -\frac{8}{9}\, x(n+2) - \frac{28}{9}\, x(n+1).$$

EHB 252E – Signals and Systems (CRN : 21012)

Midterm Examination I

21.03.2013

4 Questions, 100 Minutes

Please Show Your Work for Full Credit!

(25 pts) 1. (a) Let $f(t)$ be the continuous-time signal given as

$$f(t) = \begin{cases} 0, & \text{for } t \leq -1, \\ 1, & \text{for } -1 < t \leq 0, \\ 1-t, & \text{for } 0 < t \leq 1, \\ 0, & \text{for } 1 \geq t. \end{cases}$$

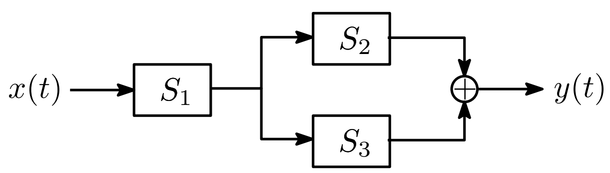
Sketch (i) $f(t)$, (ii) $f(-t)$, (iii) $f(1.5\,t)$.

(b) Let $g(n)$ be the discrete-time signal given as

$$g(n) = \begin{cases} |n|, & \text{for } -2 \leq n \leq 2, \\ 0, & \text{for } 2 < |n|. \end{cases}$$

Sketch (i) $g(n)$, (ii) $g(n-2)$, (iii) $g(2n)$.

(25 pts) 2. Consider the system S given below.



Suppose that the systems S_1, S_2, S_3 are linear time-invariant with impulse responses given as

$$h_1(t) = u(t) - u(t-2),$$
$$h_2(t) = u(t) - u(t-1),$$
$$h_3(t) = \delta(t-1),$$

where $u(t)$ is the unit step function defined as

$$u(t) = \begin{cases} 1, & \text{if } 0 \leq t, \\ 0, & \text{if } t < 0. \end{cases}$$

(a) Sketch $h_1(t)$, $h_2(t)$, $h_3(t)$.

(b) Determine the output if we input $x(t) = 1$ to the system.

(c) Determine the impulse response of the system S.

(25 pts) 3. Let T be a linear discrete-time system. Suppose that, for an integer value of k, if we input $\delta(n-k)$ to T, the output is

$$y_k(n) = k\left(\delta(n-k) + \delta(n-k-1)\right).$$

(a) Determine and sketch the output if we input $x_1(n) = 2\,\delta(n-1)$ to the system.

(b) Determine and sketch the output if we input $x_2(n) = u(n-3) - u(n)$ to the system.

(25 pts) 4. Let T be a linear time-invariant (LTI) system. Also, suppose that, if we input $x(t) = e^{j\omega t}$ to T, we observe at the output $y(t) = H(\omega)\, e^{j\omega t}$, where $H(\omega) = 1/|\omega|^2$.

(a) Suppose we input $x_1(t) = \cos(\pi\, t)$ to T. Determine the output $y_1(t)$.

(b) Suppose we input $x_2(t) = 2\,\sin(2\,\pi\, t)$ to T. Determine the output $y_2(t)$.

EHB 252E – Signals and Systems (CRN : 21012)

Midterm Examination II

02.05.2013

4 Questions, 120 Minutes

Please Show Your Work for Full Credit!

(25 pts) 1. (a) Let $f(t)$ be a continuous-time signal given as $f(t) = e^{-st}\, u(t)$, where s is a positive real number and $u(t)$ is the continuous-time unit step signal. Find $F(\omega)$, the Fourier transform of $f(t)$.

(b) Let S be a continuous-time, causal, linear time-invariant (LTI) system whose input $x(t)$ and output $y(t)$ satisfy the following differential equation :

$$y'(t) + \pi\, y(t) = 2\, x(t).$$

(i) Find $H(\omega)$, the frequency response of S.

(ii) Find $h(t)$, the impulse response of S.

(25 pts) 2. Let $x(n)$ be a discrete-time signal whose discrete-time Fourier transform (DTFT) is given by

$$X(e^{j\omega}) = \begin{cases} 1, & \text{if } |\omega| \le B, \\ 0, & \text{if } B < |\omega| \le \pi. \end{cases}$$

(a) Sketch $X(e^{j\omega})$, for $|\omega| \le \pi$.

(b) Find $x(n)$.

(c) Let $y(n)$ be a discrete-time signal given as,

$$y(n) = \begin{cases} \frac{\sin\left(\frac{\pi}{2}\, n\right)}{n}, & \text{for } n \ne 0, \\ \frac{\pi}{2}, & \text{for } n = 0. \end{cases}$$

Find $Y(e^{j\omega})$, the DTFT of $y(n)$.

(d) Compute

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

(Hint : You can make use of $y(n)$.)

(25 pts) 3. Let $x(n)$ be a discrete-time signal given as,

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{3}\right)^n u(n),$$

where $u(n)$ denotes the discrete-time unit step signal.

(a) Find $X(e^{j\omega})$, the discrete-time Fourier transform (DTFT) of $x(n)$.

(b) Find a discrete-time signal $h(n)$ such that

$$x(n) * h(n) = \delta(n),$$

where $\delta(n)$ denotes the discrete-time impulse signal.

(25 pts) 4. Let $h(t)$ be a continuous-time signal of the form

$$h(t) = \sum_{k=-\infty}^{\infty} a_k \, z(t - 2\,k),$$

where $z(t)$ is defined as,

$$z(t) = u(t+1) - u(t-1).$$

Here, $u(t)$ denotes the continuous-time unit step signal. Suppose we form the discrete-time signal $y(n)$ by sampling $h(t)$ with a period equal to $T = 2$. That is, $y(n) = h(2\,n)$ for any integer n. Suppose also that the DTFT of $y(n)$ is given as,

$$Y(e^{j\omega}) = 2 + 2\cos(\omega) - j\,\sin(2\omega).$$

(a) Sketch $z(t)$.

(b) Determine and sketch $y(n)$.

(c) Determine and sketch $h(t)$.

EHB 252E – Signals and Systems
(CRN : 21012, Instructor : İlker Bayram)

Final Examination

27.05.2013

5 Questions, 120 Minutes

Please Show Your Work for Full Credit!

(15 pts) 1. Let $g(t)$ be a continuous-time signal defined as,

$$g(t) = \begin{cases} 2+2t, & \text{for } -1 < t \le 0, \\ 2-2t, & \text{for } 0 < t \le 1, \\ 0, & \text{for } 1 \le |t|. \end{cases}$$

(a) Sketch $g(t)$.

(b) Let $h(t) = g(t) * g(t)$. Evaluate

$$c = \int_{-\infty}^{\infty} h(t)\, dt.$$

(20 pts) 2. Let S be a time-invariant system that maps a discrete-time signal x to a discrete-time signal y. Also let,

$$y(0) = 2\,x(-1) - x(0) + 3\,x(1),$$

for any input $x(n)$. Suppose $x(n) = \left(\frac{1}{2}\right)^n u(n)$ is input to the system.

(a) Find $y(0)$.

(b) Find $y(3)$.

(25 pts) 3. Let $f(t)$ be a periodic signal with period equal to $T = 2$, and

$$f(t) = \begin{cases} 0, & \text{for } -1 \le t < 0 \\ 1, & \text{for } 0 \le t \le \frac{1}{2}, \\ 0, & \text{for } \frac{1}{2} < t \le 1. \end{cases}$$

Since $f(t)$ is periodic with $T = 2$, it has a Fourier series representation

$$f(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{-j\,\pi\,k}$$

for some a_k.

(a) Find a_0.

(b) Find a_k for $k \neq 0$.

(c) Let

$$y(t) = \int_{-1}^{1} f(\tau)\, f(t-\tau)\, d\tau.$$

Observe that $y(t)$ is also periodic with $T = 2$. Therefore,

$$y(t) = \sum_{k=-\infty}^{\infty} b_k \, e^{-j\,\pi\,k}$$

for some b_k. Find b_k.

(25 pts) 4. Let $f(t)$ be a continuous-time signal whose Fourier transform is given as,

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega| < \frac{4\pi}{3}, \\ 0, & \text{for } |\omega| \geq \frac{4\pi}{3}. \end{cases}$$

Suppose we form the discrete-time signal $y(n)$ by sampling $f(t)$ with a period equal to $T = 1$. That is, $y(n) = f(n)$ for any integer n. Also let $Y(e^{j\omega})$ denote the DTFT of $y(n)$.

(a) Determine and sketch $Y(e^{j\omega})$ for $|\omega| \leq \pi$.

(b) Let $h(t) = f(t)\, e^{j\pi t}$. Also suppose we form the discrete-time signal $z(n)$ by sampling $h(t)$ with a period equal to $T = 1$. That is, $z(n) = h(n)$ for any integer n. Determine and sketch $Z(e^{j\omega})$, the DTFT of $z(n)$, for $|\omega| \leq \pi$.

(15 pts) 5. Let $x(n)$ be a discrete-time signal defined as

$$x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n),$$

where $u(n)$ denotes the discrete-time unit step signal. Also let $X(z)$ denote the z-transform of the signal.

(a) Determine $X(z)$.

(b) Find the poles and zeros of $X(z)$.

(c) Specify the region of convergence (ROC) of $X(z)$.