On the Frame Bounds of Iterated Filter Banks

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Abstract

We investigate the frame bounds of iterated non-perfect reconstruction filter banks. We provide frame bounds valid for iterated FBs with an arbitrary number of stages using the frame bounds of the underlying frame on the real line. Conversely, given the frame bounds of the iterated FB, we derive bounds for the underlying wavelet frame.

Key words: Iterated filter bank, wavelet filter bank, frame bounds.

1 Introduction

Transform based discrete-time signal processing applications typically employ an analysis filter bank (FB), a processing step operating on the output of the analysis filter bank and a synthesis filter bank, as illustrated in Figure 1. Here, the analysis and synthesis filter banks may be regarded as the analysis and synthesis operators of an underlying frame for $l_2(\mathbb{Z})$. Usually, it is desired that one be able to, at least approximately, reconstruct the input, in the absence of a processing step. In other words, it is desired that the FBs employ tight, or at least snug frames (in which cases we will call the FB a tight or snug FB). A particular type of FB that has received interest, due to its relation to wavelet frames, is the iterated filter bank, obtained by iterating an FB on its lowpass channel (see for example Figure 3). In general, conditions that ensure an FB is tight are equality constraints on the filter coefficients, which imply rather 'thin' solution sets. This has the undesirable consequence that tightness is sometimes incompatible with other requests. For example, for 2-channel critically sampled FBs (like the one shown in Figure 3), there are no real-valued, symmetric FIR solutions other than the Haar FB. Another example is regarding the double-density FB, shown in Figure 2 (see [10] for a discussion). This FB, despite its advantages over the conventional critically sampled FB, cannot be a tight frame if FIR filters are used. These motivate the use of snug iterated FBs in certain applications. However, even though snug FB design is a well-studied subject (see [6,12]), it is not clear whether the snugness of the FB would be preserved under iterations. This letter provides a partial answer to this question. In particular, we show a relation between the frame bounds of dyadic iterated FBs and dyadic wavelet frames. This relation allows one to deduce (non-optimal) frame bounds for the iterated FB (with an arbitrary number of stages) from a knowledge of the frame bounds of the underlying frame on the real line and vice versa.

To be more precise, consider an iterated FB as in Figure 3. Suppose that the FB in the dashed rectangle is orthonormal (i.e. $\{h(n-2k)\}_{k\in\mathbb{Z}} \cup \{g(n-2k)\}_{k\in\mathbb{Z}}$

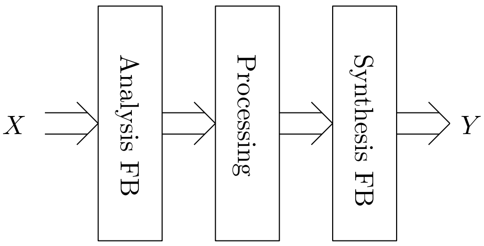

Fig. 1. Transform based discrete time signal processing applications usually employ an analysis filter bank, a processing step on the analysis FB coefficients followed by a synthesis filter bank.

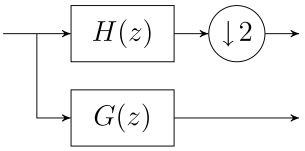

Fig. 2. This filter bank can not generate a tight frame for $l_2(\mathbb{Z})$ if FIR filters are used (also see [10] for a further discussion).

is an orthonormal basis for $l_2(\mathbb{Z})$). In this case, it follows that the iterated FB is also orthonormal, regardless of how many times it is iterated and it can be regarded as the analysis operator of an orthonormal basis, the elements of which are determined by the number of stages (see Section 3.1). On the other hand, if the FB is not orthonormal, the underlying basis for the iterated FB will not be orthonormal either. In this case, given the number of stages, along with the filters, we can compute the frame bounds of this basis. However, since the iterated FB is a rather delicate function of the number of stages (see Section 3.1), it is not clear how the frame bounds evolve when the FB is iterated. In particular, Stanhill and Zeevi [11] have given non-optimal frame bounds where the bound ratio grows exponentially with the number of stages. This might suggest (or at least does not rule out the possibility) that the frame bounds of the bases associated with iterated FBs deteriorate under iteration. Nevertheless, Stanhill and Zeevi also demonstrated through numerical examples that the optimal frame bounds do not get looser beyond some point, indicating a possible convergence. Our result provides an explanation for that.

Continuing our example, let $\phi(t)$ and $\psi(t)$ be the scaling function and wavelet associated with the filters $h(n)$ and $g(n)$ (defined by (7), (8)). We will show that if $\{\phi(t-k)\}_{k\in\mathbb{Z}} \cup \{2^{n/2}\,\psi(2^n t-k)\}_{n\in\mathbb{N},k\in\mathbb{Z}}$ is a snug Riesz basis for $L_2(\mathbb{R})$, then the iterated FB is also a snug Riesz basis for $l_2(\mathbb{Z})$ and vice versa. The Riesz basis $\{\phi(t-k)\}_{k\in\mathbb{Z}} \cup \{2^{n/2}\,\psi(2^n t-k)\}_{n\in\mathbb{N},k\in\mathbb{Z}}$ is shift-invariant like the quasi-affine system introduced by Ron and Shen [9] in order to study the frame properties of the wavelet basis ($\{2^{n/2}\,\psi(2^n t-k)\}_{n\in\mathbb{Z},k\in\mathbb{Z}}$). However, unlike the quasi-affine system, we will show that the aforementioned frame is ‘looser’ than the wavelet frame (see Lemma 10). Despite this, it lends itself more easily for the investigation of the frame bounds of iterated FBs. We will also provide a generalization for overcomplete wavelet frames by slightly modifying the arguments.

2 Preliminary and Notation

A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ s.t.

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1}$$

In this case, B and A are called the *upper* and *lower frame bounds* respectively. The particular A, B pair that minimize $B-A$ is said to be *optimal*. The frame is *tight* if $A = B$, *snug* if $A \approx B$.

For a frame $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, the operator $F : \mathcal{H} \to l_2$, defined as

$$F\,f = \{\langle f, f_k\rangle\}_{k=1}^{\infty}\,. \tag{2}$$

is called the *analysis operator*. The adjoint of this operator $F^* : l_2 \to \mathcal{H}$, called the *synthesis operator* is given by

$$F^*\,\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k\, f_k \tag{3}$$

More details can be found in [3,4]. We also refer to [2,5] for discussions on filter banks viewed as frames for $l_2(\mathbb{Z})$.

For $f(t) \in L_2(\mathbb{R})$, $\hat{f}(\omega)$ denotes its Fourier transform, where for $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, we use the definition,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt. \tag{4}$$

We also define the dilation and translation operators D^j and T^k as,

$$D^j f(t) = 2^{j/2}\, f(2^j t), \tag{5}$$
$$T^k f(t) = f(t-k), \tag{6}$$

for $j, k \in \mathbb{Z}$. For sets of functions we use $D^j\{f_i\}_{i\in\Lambda} = \{D^j f_i\}_{i\in\Lambda}$.

For a discrete-time function $h(n)$, $H(z)$ denotes its z-transform given by $H(z) = \sum_n h(n)\, z^{-n}$.

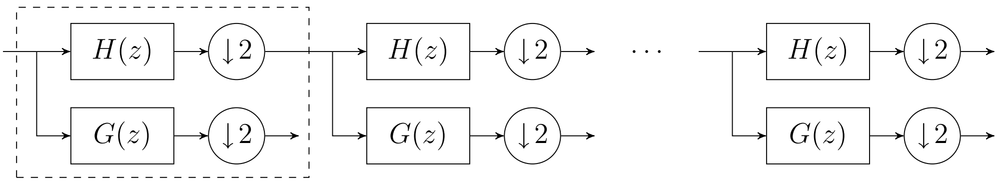

Fig. 3. A critically sampled FB obtained by iterating the lowpass branch of a critically sampled two channel FB.

We remark that if $h(n)$ is the convolution of $h_1(n)$ and $h_2(n)$ (i.e. $h(n) = \sum_k h_1(k)h_2(n-k)$), then $H(z) = H_1(z)\, H_2(z)$.

Also if $h_1(n) = h_2(Mn)$, then $H_1(z) = H_2(z^M)$. For further details see [12].

Throughout the letter, we assume that all of the discrete-time sequences are real valued.

3 From the Frame for $L_2(\mathbb{R})$ to the Iterated Filter Bank

In this section we will show that the frame bounds of the underlying frame on the real line implies bounds on the frame bounds of the iterated FB with arbitrary stages. We distinguish between the critically sampled and oversampled cases. This is due to the fact that it is only for critically sampled FBs that the underlying frame on the real line is a Riesz basis, which thus has an analysis operator mapping $L_2(\mathbb{R})$ *onto* $l_2(\mathbb{Z})$. We start with critically sampled FBs.

3.1 Critically Sampled Filter Banks

Suppose we are given the filters $h(n)$, $g(n)$ for the FB in Figure 3. We define the scaling function and wavelet as,

$$\phi(t) = \sqrt{2} \sum_{n\in\mathbb{Z}} h(n)\phi(2t-n), \tag{7}$$

$$\psi(t) = \sqrt{2} \sum_{n\in\mathbb{Z}} g(n)\phi(2t-n), \tag{8}$$

The iterated FB with m stages is equivalent to the $m+1$ channel FB in Figure

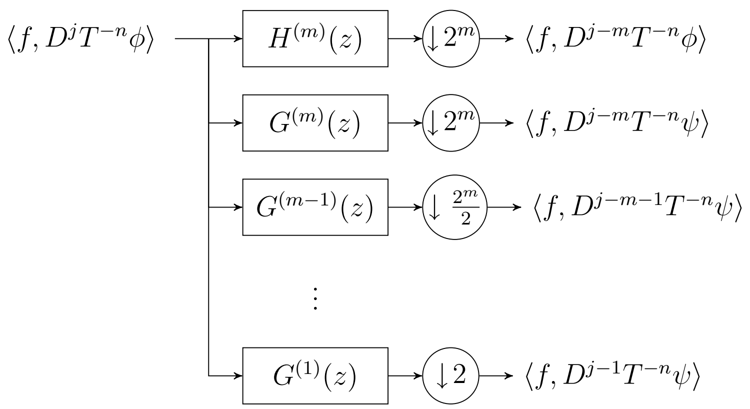

Fig. 4. The iterated FB in Figure 3 with m stages is equivalent to the $m+1$ channel FB above.

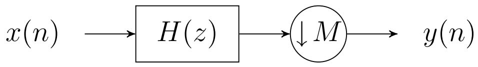

Fig. 5. This system computes $\langle x(\cdot), \{h(Mn-\cdot)\}_{n\in\mathbb{Z}}$.

4, where

$$H^{(0)}(z) = 1, \tag{9}$$
$$H^{(k)}(z) = H^{(k-1)}(z)H(z^{2^{k-1}}), \tag{10}$$
$$G^{(k)}(z) = H^{k-1}(z)G(z^{2^{k-1}}). \tag{11}$$

These can be shown using noble identities [12].

To understand the action of this FB on the input, consider a system as shown in Figure 5, consisting of a filter followed by a downsampler. The output of this system can be written as,

$$y(n) = \sum_{k\in\mathbb{Z}} x(k)\, h(Mn-k) = \langle x(\cdot), h(Mn-\cdot)\rangle. \tag{12}$$

In words, the system may be regarded as a device that computes the inner products of the input with $\{h(Mk-\cdot)\}_{k\in\mathbb{Z}}$.

To that end, the FB in Figure 4 computes the inner products of an input $x(n) \in l_2(\mathbb{Z})$ with $\{h^{(m)}(2^mk-n)\}_{k\in\mathbb{Z}}$, $\{g^{(j)}(2^jk-n)\}_{j=1,k\in\mathbb{Z}}^m$. It can be shown that provided $\{h(2k-\cdot)\}_{k\in\mathbb{Z}} \cup \{g(2k-\cdot)\}_{k\in\mathbb{Z}}$ is a Riesz basis for $l_2(\mathbb{Z})$, then so is $\{h^{(m)}(2^mk-\cdot)\}_{k\in\mathbb{Z}} \cup \{g^{(j)}(2^jk-\cdot)\}_{j=1,k\in\mathbb{Z}}^m$, albeit with different bounds. [1] In that sense, the m-channel FB in Figure 4 (or equivalently the FB in Figure 3 with m stages) can be regarded as the analysis operator for $\{h^{(m)}(2^mk-\cdot)\}_{k\in\mathbb{Z}} \cup \{g^{(j)}(2^jk-\cdot)\}_{j=1,k\in\mathbb{Z}}^m$. We denote this operator by $\mathcal{F}_m : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$.

[1] For fixed m, the optimal frame bounds can be computed by an eigenanalysis of the corresponding polyphase matrix evaluated on the unit circle – see [2].

For $f \in L_2(\mathbb{R})$, it is well-known that (see for example [7]), if we input $\langle f, D^j T^{-n}\phi\rangle$ to the m-stage iterated FB (the discrete time variable being 'n'), then the low-pass channel outputs $\langle f, D^{j-m}T^{-n}\phi\rangle$, and the k^{th} bandpass channel outputs $\langle f, D^{j-k}T^{-n}\psi\rangle$ (see Figure 4). In this case, we can write,

$$\|\mathcal{F}_m\langle f, D^j T^{-n}\phi\rangle\|^2 = \sum_{n\in\mathbb{Z}} |\langle f, D^{j-m}T^{-n}\phi\rangle|^2 + \sum_{k=j-m}^{j-1}\sum_{n\in\mathbb{Z}} |\langle f, D^k T^{-n}\psi\rangle|^2. \quad (13)$$

Given these definitions, the main result of this subsection is,

Theorem 1 *If $\{T^k\,\phi\}_{k\in\mathbb{Z}} \cup \{D^j\,T^k\,\psi\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ is a Riesz basis for $L_2(\mathbb{R})$ with bounds A, B, then the iterated FB is a Riesz basis for $l_2(\mathbb{Z})$ with bounds A/B, B/A, regardless of the number of stages.*

We will use the following lemma.

Lemma 2 *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a frame for $L_2(\mathbb{R})$. $\{f_k\}_{k\in\mathbb{Z}}$ has the same frame bounds as $D^m\{f_k\}_{k\in\mathbb{Z}}$.*

Proof of Thm. 1: Since $D^m\left\{\{T^k\,\phi\}_{k\in\mathbb{Z}} \cup \{D^j\,T^k\,\psi\}_{j\in\mathbb{N},k\in\mathbb{Z}}\right\}$ is a Riesz basis (by Lemma 2), its analysis operator maps $L_2(\mathbb{R})$ *onto* $l_2(\mathbb{Z})$. As such, suppose we are given an arbitrary $x(n) \in l_2(\mathbb{Z})$. We can find $f \in L_2(\mathbb{R})$ s.t.

$$\langle f, D^m\,T^{-n}\phi\rangle = x(n), \quad (14)$$

$$\langle f, D^{m+r}\,T^{-n}\psi\rangle = 0 \quad \text{for } r \geq 0. \quad (15)$$

Again by Lemma 2, $D^m\left\{\{T^k\,\phi\}_{k\in\mathbb{Z}} \cup \{D^j\,T^k\,\psi\}_{j\in\mathbb{N},k\in\mathbb{Z}}\right\}$ has the frame bounds A, B. Thus,

$$A \leq \frac{\|x(n)\|^2}{\|f\|^2} \leq B. \quad (16)$$

Now notice that

$$\sum_n |\langle f, T^{-n}\phi\rangle|^2 + \sum_{k=0}^{m-1}\sum_{n\in\mathbb{Z}} |\langle f, D^k\,T^{-n}\psi\rangle|^2 = \|\mathcal{F}_m x(n)\|^2. \quad (17)$$

Noting the frame bounds of $\{T^k\,\phi\}_{k\in\mathbb{Z}} \cup \{D^j\,T^k\,\psi\}_{j\in\mathbb{N},k\in\mathbb{Z}}$, this implies

$$A \leq \frac{\|\mathcal{F}_m\,x(n)\|^2}{\|f\|^2} = \frac{\|\mathcal{F}_m\,x(n)\|^2}{\|x(n)\|^2}\frac{\|x(n)\|^2}{\|f\|^2} \leq B, \quad (18)$$

Noting (16), we get,

$$\frac{A}{B} \leq \frac{\|\mathcal{F}_m x(n)\|^2}{\|x(n)\|^2} \leq \frac{B}{A} \quad (19)$$

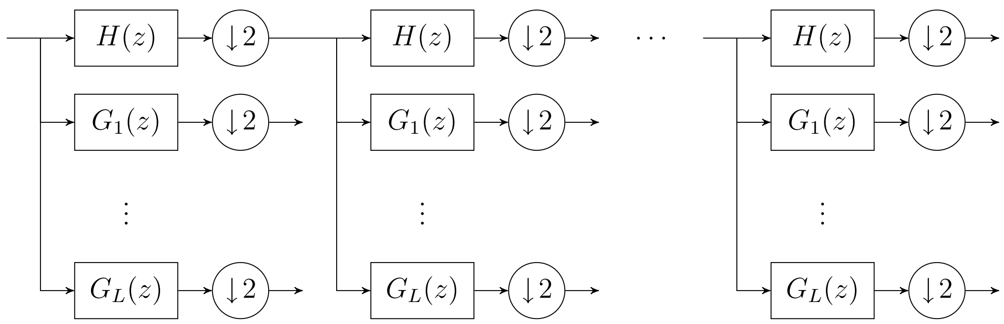

Fig. 6. An overcomplete FB obtained by iterating the lowpass branch of an overcomplete $L+1$ channel FB.

By the arbitrariness of $x(n)$, it follows that the iterated FB with m stages has the frame bounds A/B, B/A. Notice that the bounds are independent of the number of stages.

$\square$

Remark 3 *The resulting bounds on the iterated FB with an arbitrary number of stages are not necessarily tight. We will return to this issue in the Section 4, where we provide a result in the converse direction. This being so, it can be derived from this theorem that if the underlying frame on the real line is orthonormal, then the FB is orthonormal too. Even though this is well known, this indicates that the provided bounds are indeed not too loose.*

3.2 Overcomplete Filter Banks

Consider now the overcomplete iterated FB in Figure 6. We will denote the analysis operator for the iterated FB with m stages as $\mathcal{F}_m$ as in the previous subsection. We define the scaling and wavelet functions as,

$$\phi(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} h(n) \phi(2t - n), \tag{20}$$

$$\psi_i(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} g_i(n) \phi(2t - n), \quad i = 1, 2, \ldots, L. \tag{21}$$

Using the scaling function, the approximation spaces are defined as,

$$V_n = D^n \, \overline{\text{span}} \{T^k \phi(t)\}_{k \in \mathbb{Z}}. \tag{22}$$

We also define

$$\Phi(\omega) = \sum_{k \in \mathbb{Z}} |\hat{\phi}(\omega + k2\pi)|^2. \tag{23}$$

We remark that $D^m \left\{ \{T^k\, \phi(t)\}_{k\in\mathbb{Z}} \cup \{D^j\, T^k\, \psi_i(t)\}_{i=1,j\in\mathbb{N},k\in\mathbb{Z}}^L \right\}$ is not a Riesz basis for $L_2(\mathbb{R})$. However, provided there exist $a, b > 0$ s.t $a < \Phi(\omega) < b$, $D^m\{T^k\phi(t)\}_{k\in\mathbb{Z}}$ *is* a Riesz frame sequence for V_m (see [1] and [3], Chp. 7) and its analysis operator maps V_m *onto* $l_2(\mathbb{Z})$.

Theorem 4 *Let $\{T^k\, \phi(t)\}_{k\in\mathbb{Z}} \cup \{D^j\, T^k\, \psi_i(t)\}_{i=1,j\in\mathbb{N},k\in\mathbb{Z}}^L$ be a frame with bounds A, B, associated with the iterated FB in Figure 6. Suppose $a < \Phi(\omega) < b$ almost everywhere where $\infty > b, a > 0$. Set*

$$\alpha = \sup_{f\in V_0} \frac{1}{\|f\|^2} \sum_{i=1}^{L} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} |\langle f, D^j T^k \psi_i\rangle|^2. \tag{24}$$

If $A > \alpha$, then the iterated FB is a frame for $l_2(\mathbb{Z})$ with frame bounds $(A-\alpha)/B$, $B/(A-\alpha)$, regardless of the number of stages.

Proof: The proof follows along the same lines as the proof of Thm 1, with the necessary adjustments.

Since $D^m\{T^k\phi(t)\}_{k\in\mathbb{Z}}$ is a Riesz frame sequence for V_m, given an arbitrary $x(n) \in l_2(\mathbb{Z})$, we can find $f \in V_m$ s.t.

$$\langle f, D^m\, T^{-n}\, \phi(t)\rangle = x(n), \tag{25}$$

By the definition of α and a scaling argument as in Lemma 2, we obtain

$$\sum_{i=1}^{L} \sum_{j=m}^{\infty} \sum_{k\in\mathbb{Z}} |\langle f, D^j\, T^k\, \psi_i\rangle|^2 \le \alpha \|f\|^2. \tag{26}$$

Since $D^m \left\{ \{T^k\, \phi(t)\}_{k\in\mathbb{Z}} \cup \{D^j\, T^k\, \psi_i(t)\}_{i=1,j\in\mathbb{N},k\in\mathbb{Z}}^L \right\}$ has the frame bounds A, B, we thus have

$$A - \alpha \le \frac{\|x(n)\|^2}{\|f\|^2} \le B. \tag{27}$$

Now notice

$$\sum_n |\langle f, T^{-n}\, \phi\rangle|^2 + \sum_{i=1}^{L} \sum_{j=0}^{m-1} \sum_{n\in\mathbb{Z}} |\langle f, D^j\, T^{-n}\, \psi_i\rangle|^2 = \|\mathcal{F}_m x(n)\|^2. \tag{28}$$

Noting the frame bounds of $\{T^k\, \phi(t)\}_{k\in\mathbb{Z}} \cup \{D^j\, T^k\, \psi_i(t)\}_{i=1,j\in\mathbb{N},k\in\mathbb{Z}}^L$, this implies

$$A - \alpha \le \frac{\|\mathcal{F}_m\, x(n)\|^2}{\|f\|^2} = \frac{\|\mathcal{F}_m\, x(n)\|^2}{\|x(n)\|^2} \frac{\|x(n)\|^2}{\|f\|^2} \le B, \tag{29}$$

Noting (27), we get,

$$\frac{A-\alpha}{B} \le \frac{\|\mathcal{F}_m\, x(n)\|^2}{\|x(n)\|^2} \le \frac{B}{A-\alpha} \tag{30}$$

□

Remark 5 *Even though this theorem provides bounds similar to Thm. 1, the bounds are somewhat looser, for when given a snug frame on the real line, the implied bounds for the iterated FB will be loose. This stems from the fact that for overcomplete FBs, the lowpass filter needs to satisfy the inequality* $|H(e^{j\omega})|^2 + |H(e^{j\omega+\pi})|^2 \leq 2$ *(see e.g. [8,10]). This in turn means that* $H(e^{j\omega})$ *is concentrated on a subset* $[-a\pi, a\pi]$ *with* $a < 1/2$*, implying that* $\hat{\phi}(\omega)$ *is concentrated on* $[-2a\pi, 2a\pi]$*. As such,* $\Phi(\omega)$ *is far from being a constant and* $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ *is not a snug frame sequence for* V_0*. Thus, in order for the overall frame* $\{T^k\phi(t)\}_{k\in\mathbb{Z}} \cup \{D^jT^k\psi_{(i)}(t)\}_{i=1,j\in\mathbb{N},k\in\mathbb{Z}}^L$ *to be snug,* α *should not be too small. Consequently,* $(A-\alpha)/B$ *and* $B/(A-\alpha)$ *cannot be very close.*

Remark 6 *Tighter bounds for the iterated FB can be obtained when attention is restricted to a subspace of* $l_2(\mathbb{Z})$*. To see this, take* $V_0' \subset V_0$*. Then we will have*

$$\alpha' = \sup_{f\in V_0'} \frac{1}{\|f\|^2} \sum_{i=1}^{L} \sum_{j\in\mathbb{N},k\in\mathbb{Z}} |\langle f, D^jT^k\psi_i\rangle|^2 < \alpha. \tag{31}$$

Thus for the subspace $X = \{x(n) : \exists f \in V_0' \text{ s.t. } x(n) = \langle f, \phi(t-n)\rangle\}$*, the iterated FB provides a frame sequence with frame bounds* $(A-\alpha')/B$*,* $B/(A-\alpha')$*.*

4 From the Iterated Filter Bank to the Wavelet Frame

In this section, we derive results in the converse direction. Given frame bounds for iterated FBs valid for an arbitrary number of stages, we will obtain frame bounds for the underlying frame on the real line. Unlike the last section, we will not need to discriminate between the overcomplete and critically sampled cases. Therefore the results are stated for overcomplete FBs, implying the same for the critically sampled case.

We adopt the definitions in Section 3.2.

Theorem 7 *If the iterated FB in Figure 6 is a frame for* $l_2(\mathbb{Z})$ *with bounds* A*,* B*, regardless of the number of stages, and* $\Phi(\omega)$ *is continuous at* $\omega = 0$ *with* $\Phi(0) = c$*, then* $\{T^k\,\phi(t)\}_{k\in\mathbb{Z}} \cup \{D^j\,T^k\,\psi_i(t)\}_{i=1,j\in\mathbb{N},k\in\mathbb{Z}}^L$ *is a frame for* $L_2(\mathbb{R})$ *with bounds* cA*,* cB*.*

As a corollary of this (and also an auxiliary result), it follows also that

Theorem 8 *If the iterated FB in Figure 6 is a frame for* $l_2(\mathbb{Z})$ *with bounds* A*,* B*, regardless of the number of stages, and* $\Phi(\omega)$ *is continuous at* $\omega = 0$ *with* $\Phi(0) = c$*, then* $\{D^j\,T^k\,\psi_i(t)\}_{i=1,j\in\mathbb{Z},k\in\mathbb{Z}}^L$ *is a frame for* $L_2(\mathbb{R})$ *with bounds* *cA, cB.*

We prove Thm. 7 by modifying the proof of the unitary extension principle given by Benedetto and Treiber [1].

Proof of Thm 7: Suppose the iterated FB has the frame bounds A, B, regardless of the number of stages. Pick $\epsilon > 0$. Let $f \in L_2(\mathbb{R})$ be given s.t. $\hat{f}$ is continuous and compactly supported. Since $\Phi(\omega)$ is continuous at 0 with $\Phi(0) = c$, we can find $N \in \mathbb{Z}$ (see for example Lemma 14.2.2 in [3]) s.t. if $m \geq N$,

$$(c-\epsilon)\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, D^m T^k \phi\rangle|^2 \leq (c+\epsilon)\|f\|^2. \tag{32}$$

We can also find $K \in \mathbb{Z}$ s.t.

$$\sum_{i=1}^{L} \sum_{j=K}^{\infty} \sum_{k\in\mathbb{Z}} |\langle f, D^j T^k \psi_i\rangle|^2 \leq \epsilon \|f\|^2 \tag{33}$$

Set $M = \max\{N, K\}$. Notice

$$\sum_{k\in\mathbb{Z}} |\langle f, T^k \phi\rangle|^2 + \sum_{i=1}^{L} \sum_{j=0}^{M-1} \sum_{k\in\mathbb{Z}} |\langle f, D^j T^k \psi_i\rangle|^2 = \|\mathcal{F}_M \langle f, D^M T^{-k} \phi\rangle\|^2. \tag{34}$$

Since $A\|x(n)\|^2 \leq \|\mathcal{F}_M x(n)\|^2 \leq B\|x(n)\|^2$, this implies that

$$A(c-\epsilon)\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, T^k \phi\rangle|^2 + \sum_{i=1}^{L} \sum_{j=0}^{M-1} \sum_{k\in\mathbb{Z}} |\langle f, D^j T^k \psi_i\rangle|^2 \leq B(c+\epsilon)\|f\|^2. \tag{35}$$

Adding (33), we get

$$A(c-\epsilon)\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, T^k \phi\rangle|^2 + \sum_{i=1}^{L} \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} |\langle f, D^j T^k \psi_i\rangle|^2 \leq (B(c+\epsilon)+\epsilon)\|f\|^2. \tag{36}$$

By the arbitrariness of ϵ and f, it follows that $\{T^k\phi(t)\}_{k\in\mathbb{Z}} \cup \{D^j T^k \psi_i(t)\}_{i=1, j\in\mathbb{N}, k\in\mathbb{Z}}^{L}$ is a frame with bounds cA, cB for the set of functions $f \in \text{Ł}_2(\mathbb{R})$ which have compactly supported and continuous Fourier transforms. Since this set is dense in $L_2(\mathbb{R})$, the theorem follows (by Lemma 5.1.7 in [3]).

□

Remark 9 *Now that we have a converse to Thm. 1, we can test whether they can be used to obtain optimal frame bounds, or not. Suppose that Thm. 7 gives the optimal bounds for the underlying frame on the real line. That is, if our FB is critically sampled and A, B are the optimal bounds valid for arbitrary number of stages, then the bounds implied for the underlying Riesz basis on* *the real line are also optimal. This implies that the bounds given in Thm 1 are not optimal. For if we start with a critically sampled FB bounded by A, B, the underlying Riesz basis has the bounds cA, cB according to Thm 7. But now Thm. 1 implies that the FB is bounded by A/B, B/A, which cannot be optimal unless $B = A = 1$. A similar argument yields that Thm. 7 cannot give optimal bounds if Thm. 1 can.*

Thm. 8 is a corollary of this theorem and the following lemma.

Lemma 10 *Let $\{T^k\phi(t)\}_{k\in\mathbb{Z}} \cup \{D^jT^k\psi_i(t)\}_{i=1,j\in\mathbb{N},k\in\mathbb{Z}}^L$ be a frame with bounds A, B. Then $\{D^jT^k\psi_i\}_{i=1,j\in\mathbb{Z},k\in\mathbb{Z}}^L$ is also a frame with bounds A, B.*

Proof: First, let us show that $\{D^jT^k\psi_i\}_{i=1,j\in\mathbb{Z},k\in\mathbb{Z}}^L$ is a Bessel sequence with bound B. Suppose this is not true. Then we can find $f \in L_2$ and $N \in \mathbb{Z}$ s.t.

$$\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, D^jT^k\psi_i\rangle|^2 > B\,\|f\|^2. \tag{37}$$

By Lemma 2, $D^N\left\{\{T^k\phi\}_{k\in\mathbb{Z}} \cup \{D^jT^k\psi_i\}_{i=1,j\in\mathbb{N},k\in\mathbb{Z}}^L\right\}$ has the upper frame bound B. Therefore,

$$B\|f\|^2 \geq \sum_{k\in\mathbb{Z}} |\langle f, D^NT^k\phi\rangle|^2 + \sum_{i=1}^{L}\sum_{j=N}^{\infty}\sum_{k\in\mathbb{Z}} |\langle f, D^jT^k\psi_i\rangle|^2 > B\,\|f\|^2, \tag{38}$$

which is a contradiction.

Now the lower bounds. Pick an ϵ with $A > 2\,\epsilon > 0$. Suppose we can find $f \in L_2(\mathbb{R})$ s.t.

$$\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, D^jT^k\psi_i\rangle|^2 < (A - 2\,\epsilon)\|f\|^2. \tag{39}$$

For this f, we can also find $N \in \mathbb{Z}$ (for a proof see part (ii) of Lemma 14.2.5 in [3], or Lemma 7.7 in [1]) s.t.

$$\sum_{k\in\mathbb{Z}} |\langle f, D^NT^k\phi\rangle|^2 < \epsilon\|f\|^2. \tag{40}$$

Once again invoking Lemma 2, we have,

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, D^NT^k\phi\rangle|^2 + \sum_{i=1}^{L}\sum_{j=N}^{\infty}\sum_{k\in\mathbb{Z}} |\langle f, D^jT^k\psi_i\rangle|^2 < (A - \epsilon)\|f\|^2, \tag{41}$$

a contradiction. By the arbitrariness of ϵ, it follows that

$$\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, D^jT^k\psi_i\rangle|^2 \geq A\,\|f\|^2, \quad \forall\, f \in L_2(\mathbb{R}). \tag{42}$$

□

Remark 11 *When the FB is perfect reconstruction (i.e. tight with frame bound equal to 1), then the iterated FB is also tight, regardless of the number of stages. According to Thm. 8, this implies that the resulting wavelet frame is also tight. This special case is in fact the unitary extension principle of Ron and Shen [9].*

5 Discussion

We showed that the knowledge of the frame bounds of the iterated FB can be used to obtain the frame bounds of the underlying frame on the real line and vice versa. This implies that an FB, possibly non-tight to start with, will not have deteriorating frame bounds as it is iterated. However, an important question regarding the design of iterated FBs remains. That is, what are the conditions, if any, on the filters (directly, that is, without referring to the scaling function or the wavelet) which will yield a non-perfect reconstruction system (but will possibly possess other useful properties) and will be stable under iterations? We hope that this letter provides some motivation towards answering this question.

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