# MAT 271E – Probability and Statistics

Spring 2013

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| Instructor : | İlker Bayram<br>EEB 1103<br>`ibayram@itu.edu.tr` |
| Class Meets : | 13.30 – 16.30, Wednesday<br>EEB 5303 |
| Office Hours : | 10.00 – 12.00, Wednesday |
| Textbook : | D. B. Bertsekas, J. N. Tsitsiklis, 'Introduction to Probability',<br>$2^{\text{nd}}$ Edition, Athena-Scientific. |
| Supp. Text : | • D. Wackerly, W. Mendenhall, R. L. Scheaffer,<br>'Mathematical Statistics with Applications'.<br>• S. Ross, 'Introduction to Probability and Statistics for Engineers and Scientists'. |
| Grading : | 2 Midterms (30% each), Final (40%). |
| Webpage : | `http://ninova.itu.edu.tr/Ders/1318/Sinif/7783` |

### Tentative Course Outline

- Probability Space
  *Probability models, conditioning, Bayes' rule, independence.*
- Discrete Random Variables
  *Probability mass function, functions of random variables, expectation, joint PMFs, conditioning, independence.*
- General Random Variables
  *Probability distribution function, cumulative distribution function, continuous Bayes' rule, correlation, conditional expectation.*
- Limit Theorems
  *Law of large numbers, central limit theorem.*
- Introduction to Statistics
  *Parameter estimation, linear regression, hypothesis testing.*

# MAT 271E – Homework 1

Due 20.02.2013

1. Let $S_1$, $S_2$, …, $S_N$ be sets.

   (a) Show that
   $$\left(\bigcap_{n=1}^{N} S_n\right)^c = \bigcup_{n=1}^{N} S_n^c.$$

   (b) Show that
   $$\left(\bigcup_{n=1}^{N} S_n\right)^c = \bigcap_{n=1}^{N} S_n^c.$$

   **Solution.** (a) Done in class.

   (b) Let us define $A = \left(\bigcup_{n=1}^{N} S_n\right)^c$, $B = \bigcap_{n=1}^{N} S_n^c$. We will show the equality $A = B$ in two steps. We will first show that $A \subset B$. Next we will show that $A \supset B$.
   Now pick $x \in A$. Note that $x \notin S_n$ for any $n$, because otherwise we would have $x \in \bigcup_{n=1}^{N} S_n = A^c$, which is not possible. This implies that $x \in S_n^c$ for all $n$. therefore, it also in the intersection of these sets, i.e. $x \in \bigcap_{n=1}^{N} S_n^c = B$. We showed that an arbitrary element of $A$ is also in $B$. Therefore, $A \subset B$.
   Now pick $x \in B$. This means that $x \in S_n^c$ for all $n$. Equivalently, we can say that $x \notin S_n$ for any $n$. Therefore, $x \notin \bigcup_{n=1}^{N} S_n$. But then, we should have $x \notin \left(\bigcup_{n=1}^{N} S_n\right)^c = A$. We showed that an arbitrary element of $B$ is also in $A$. Therefore, $A \supset B$.

2. (a) Suppose $S$ is a set with a finite number of elements. Suppose also that the number of distinct subsets of $S$ is equal to $N_S$. Also, let $x \notin S$ and $D = \{x\} \cup S$. Now let $N_D$ be the number of distinct subsets of $D$. Express $N_D$ in terms of $N_S$.

   (b) Let $m(n)$ denote the number of subsets of a set with $n$ elements. Express $m(n+1)$ in terms of $m(n)$.

   (c) For the functions in part (b), what is $m(1)$? Find an expression for $m(n)$.

   **Solution.** (a) Let $C_1, C_2, \ldots, C_{N_S}$ denote the subsets of $S$ (observe that there are $N_S$ of them in total, $\emptyset$ and $S$ are among these sets). Given these subsets, note that the subsets of $D$ can be listed as $C_1, C_2, \ldots, C_{N_S}, \big(\{x\} \cup C_1\big), \big(\{x\} \cup C_2\big), \ldots, \big(\{x\} \cup C_{N_S}\big)$ (Why?). From this, we observe that the number of subsets of $D$ is $N_D = 2\,N_S$.

   (b) From (a), we have, $m(n+1) = 2\,m(n)$.

   (c) Since the subsets of a set with a single element are the (i) set itself and (ii) the empty set, we have $m(1) = 2$. So,
   $$m(n) = 2\,m(n-1) = 2^2\,m(n-2) = \ldots = 2^{n-1}\,m(1) = 2^n.$$

3. In an election with three candidates, $C_1$, $C_2$, $C_3$, let $A_i$ denote the event that $C_i$ wins the election. Suppose we are given the following probabilities.
   $$P\big(A_1^c\big) = 0.7, \quad P\big(A_2^c\big) = 0.4.$$
   Find the probability that $C_1$ or $C_2$ wins the election.

   **Solution.** Let $B$ be the event $B = \{C_1 \text{ or } C_2 \text{ wins the election}\}$ . Observe that $B$ can equivalently be expressed as $B = \{C_3 \text{ loses the election}\}$. In terms of $A_i$'s, this is, $B = A_3^c$. Thus, we need to find $P(A_3^c)$.

Observe that $A_i \cap A_j = \emptyset$ if $i \neq j$ (if a candidate wins, the others must lose). Also, $A_1 \cup A_2 \cup A_3 = \Omega$, where $\Omega$ denotes the sample space (one of them will definitely win). Therefore,

$$P(A_1) + P(A_2) + P(A_3) = 1.$$

From the given information, we find that $P(A_1) = 1 - P(A_1^c) = 0.3$, $P(A_2) = 1 - P(A_2^c) = 0.6$. Thus,

$$P(A_3) = 1 - P(A_1) - P(A_2) = 0.1.$$

From this we find that $P(B) = P(A_3^c) = 1 - P(A_3) = 0.9$.

4. Let $\Omega$ be a sample space, and $A$, $B$, two events from this sample space. Also, let $P$ denote a probability law which satisfies the axioms of probabity. Recall that $P$ assigns a probability to each event in the sample space. Show that $P\big(A \cup B\big) = P(A) + P(B) - P\big(A \cap B\big)$.

   **Solution.** Observe that we can decompose the sets as

$$\begin{aligned} A &= \big(A \cap B^c\big) \cup \big(A \cap B\big), \\ B &= \big(B \cap A^c\big) \cup \big(A \cap B\big), \\ A \cup B &= \big(A \cap B^c\big) \cup \big(B \cap A^c\big) \cup \big(A \cap B\big), \end{aligned}$$

   where on each line, the sets enclosed in parentheses have empty intersections (i.e., $\big(A \cap B^c\big) \cap \big(A \cap B\big) = \emptyset$, etc.) (you can draw Venn diagrams to convince yourself). By the additivity propery of a probability law, we therefore have,

$$\begin{aligned} P(A) &= P\big(A \cap B^c\big) + P\big(A \cap B\big), \\ P(B) &= P\big(B \cap A^c\big) + P\big(A \cap B\big), \\ P(A \cup B) &= P\big(A \cap B^c\big) + \big(B \cap A^c\big) + \big(A \cap B\big), \end{aligned}$$

   Summing the first two equalities above, we obtain

$$P(A) + P(B) = \underbrace{P\big(A \cap B^c\big) + P\big(A \cap B\big) + P\big(B \cap A^c\big)}_{P(A \cup B)} + P\big(A \cap B\big).$$

# MAT 271E – Homework 2

Due 27.02.2013

1. An urn contains 5 white, 10 black balls. Your friend draws a ball at random, looks at its color, and then puts 2 balls of that color back into the urn (so that there are now 16 balls in total). Now you draw a ball randomly. What is the probability that the ball you draw is black?

   **Solution.** Let us define the events,

   $$W = \{\text{your friend draws a white ball}\},$$
   $$B = \{\text{you draw a black ball}\}.$$

   We are asked to compute $P(B)$. This is easier to compute if we condition $B$ on $W$ and $W^c$.

   $$P(B) = P(B|W)\,P(W) + P(B|W^c)\,P(W^c).$$

   Observe that $P(W) = 1 - P(W^c) = 5/15$. Also, $P(B|W) = 10/16$, $P(B|W^c) = 11/16$. Therefore,

   $$P(B) = \frac{10}{16}\,\frac{1}{3} + \frac{11}{16}\,\frac{2}{3}.$$

2. You toss a coin $n$ times. Assume that the coin is biased so that the probability of observing a Head is equal to $p$. Suppose that the probability of observing $k$ Heads in $n-1$ tosses is given by $P_{n-1}(k)$ for $k = 0, 1, \ldots, n-1$ (observe that this is a function of $k$). Now let $P_n(k)$ denote the probability of observing $k$ Heads in $n$ tosses.

   (a) Express $P_n(0)$ in terms of $P_{n-1}(0)$, $P_{n-1}(1)$, $\ldots$, $P_{n-1}(n-1)$.

   (b) Express $P_n(k)$ (where $0 < k \leq n$) in terms of $P_{n-1}(0)$, $P_{n-1}(1)$, $\ldots$, $P_{n-1}(n-1)$.

   (Hint : Use the Total Probability Theorem.)

   **Solution.** (a) Observing 0 Heads in $n$ tosses is possible only if we observe $n-1$ Tails in the first $n-1$ tosses and the $n^{\text{th}}$ toss is also a Head. Thus,

   $$P_n(0) = (1-p)\,P_{n-1}(0).$$

   (b) Observing $k$ Heads in $n$ tosses is possible if either

   (i) we observe $k-1$ Heads in the first $n-1$ tosses and the $n^{\text{th}}$ toss is a Head,

   (ii) we observe $k$ Heads in the first $n-1$ tosses and the $n^{\text{th}}$ toss is a Tail.

   Notice that the two events described in (i) and (ii) are distinct. That is, their intersection is the impossible event, $\emptyset$. We compute the probability of the event in (i) as $p\,P_{n-1}(k-1)$. The probability of the event in (ii) is $(1-p)\,P_{n-1}(k)$. Therefore, we obtain, for $0 < k \leq n$,

   $$P_n(k) = p\,P_{n-1}(k-1) + (1-p)\,P_{n-1}(k).$$

3. An urn contains $w$ white balls and $b$ black balls. You know that $k_w$ of the white balls and $k_b$ of the black balls are marked with a red dot. Your friend draws a ball at random and tells you that the ball is marked. What is the probability that the drawn ball is white?

   **Solution.** Let us define the events,

   $$W = \{\text{a white ball is drawn}\},$$
   $$B = \{\text{a black ball is drawn}\},$$
   $$M = \{\text{a marked ball is drawn}\}.$$

   In terms of these events, we are asked to compute $P(W|M)$. We can write

   $$P(W|M) = \frac{P(W \cap M)}{P(M)} = \frac{P(M|W)\,P(W)}{P(M|W)\,P(W) + P(M|W^c)\,P(W^c)}.$$

   Notice that $W^c = B$. The probabilities above are given as,

   $$P(M|W) = \frac{k_w}{w}, \quad P(M|W^c) = \frac{k_b}{b}, \quad P(W) = \frac{w}{w+b}, \quad P(W^c) = \frac{b}{w+b}.$$

# MAT 271E – Homework 3

Due 06.03.2013

1. You roll a fair die 10 times. Assume that the rolls are independent. What is the probability that you observe a '1' at least twice.

   **Solution.** Let us define the events,

   $$\begin{aligned} A &= \{\text{we observe 1 at least twice}\}, \\ B_1 &= \{\text{we observe 1 once}\}, \\ B_0 &= \{\text{we do not observe any 1's}\}. \end{aligned}$$

   We are asked to compute $P(A)$. Notice that $A = \big(B_0 \cup B_1\big)^c$. Therefore, $P(A) = 1 - P(B_0 \cup B_1)$. Observe also that $B_0 \cap B_1 = \emptyset$. Therefore, $P(B_0 \cup B_1) = P(B_0) + P(B_1)$. Since the die is fair and the rolls are independent, we have

   $$P(B_0) = \left(\frac{5}{6}\right)^{10}, \quad P(B_1) = 10\,\frac{1}{6}\left(\frac{5}{6}\right)^{9}.$$

   Therefore,

   $$P(A) = 1 - \frac{15}{6}\left(\frac{5}{6}\right)^{9}.$$

2. You roll a fair die until you observe a '1'. Assume that the rolls are independent. What is the probability that you roll at least 6 times?

   **Solution.** For this experiment, consider the events,

   $$\begin{aligned} A &= \{\text{we roll at least 6 times}\}, \\ B &= \{\text{we do not observe a 1 in the first five rolls}\}. \end{aligned}$$

   Notice that $A$ and $B$ are equivalent events. Therefore,

   $$P(A) = P(B) = \left(\frac{5}{6}\right)^{5}.$$

3. You roll a fair die until you observe a '1' twice. Assume that the rolls are independent. What is the probability that you roll at least 6 times?

   **Solution.** For this experiment, consider the events,

   $$\begin{aligned} A &= \{\text{we roll at least 6 times}\}, \\ B_0 &= \{\text{we do not observe a 1 in the first five rolls}\}, \\ B_1 &= \{\text{we observe only a single 1 in the first five rolls}\}. \end{aligned}$$

   Notice that $A = B_0 \cup B_1$. Moreover, $B_0 \cap B_1 = \emptyset$. Therefore, $P(A) = P(B_0) + P(B_1)$. Noting that

   $$P(B_0) = \left(\frac{5}{6}\right)^{5}, \quad P(B_1) = 5\,\frac{1}{6}\left(\frac{5}{6}\right)^{4},$$

   we obtain,

   $$P(A) = 2\left(\frac{5}{6}\right)^{5}.$$

4. You roll a fair die until you observe a '1' twice. Assume that the rolls are independent. What is the probability that you observe a '6' at least once?

**Solution.** Given this experiment, let us define the event,

$$A = \{\text{we observe a '6' at least once}\}.$$

Notice that

$$A^c = \{\text{we do not observe a '6'}\}.$$

Suppose we somehow know that we (will) roll $n$ times. Let us call this event $R_n$. That is,

$$R_n = \{\text{we roll } n \text{ times}\}.$$

Notice that this is only possible if the second '1' occurs at the $n^{\text{th}}$ roll. Therefore, for $n > 2$, we will have,

$$P(A^c|R_n) = (n-1)\,\frac{1}{6}\,\left(\frac{4}{6}\right)^{n-2}.$$

Observe also that $R_n \cap R_k = \emptyset$ for any $(n,k)$ pair with $n \neq k$. In addition, we have $\cup_{i=2}^{\infty} R_n = \Omega$. Thus, $R_2, R_3, \ldots$ forms a partition of $\Omega$. Therefore, we can write

$$P(A^c) = \sum_{i=2}^{\infty} P(A^c|R_n)\,P(R_n).$$

Note that,

$$P(R_n) = (n-1)\,\frac{1}{6}\,\left(\frac{5}{6}\right)^{n-2}.$$

Thus,

$$P(A^c) = \sum_{i=2}^{\infty} (n-1)^2 \left(\frac{1}{6}\right)^2 \left(\frac{4}{6}\right)^{n-2} \left(\frac{5}{6}\right)^{n-2}, \tag{1}$$

and

$$P(A) = 1 - P(A^c).$$

**Note** : You can actually evaluate the sum in (1). For this, consider the function

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

where $|x| < 1$. Differentiate $f(x)$ and see what happens.

# MAT 271E – Homework 4

Due 13.03.2013

1. There are 3 white and 5 black balls in an urn. You draw two balls from the urn randomly (without replacement). Let $X$ be a random variable defined to be equal to the number of white balls drawn. What is the probability mass function (PMF) of $X$?

   **Solution.** Note that the sample space for this experiment can be written as,

   $$\Omega = \{WW, WB, BW, BB\},$$

   where $BW$ corresponds to the outcome where the first ball is black and the second one is white, etc. In this case, observe that

   $$\begin{aligned} p_X(0) &= P(\{X = 0\}) = P(\{BB\}) = \frac{5}{8}\frac{4}{7}, \\ p_X(1) &= P(\{X = 1\}) = P(\{BW, WB\}) = \frac{5}{8}\frac{3}{7} + \frac{3}{8}\frac{5}{7}, \\ p_X(2) &= P(\{X = 2\}) = P(\{WW\}) = \frac{3}{8}\frac{2}{7}. \end{aligned}$$

   For values of $k$ other than 0,1,2, we will have $p_X(k) = 0$.

2. Let $X$ be a 'geometric random variable' with a distribution given as

   $$p_X(k) = \begin{cases} \left(\frac{1}{2}\right)^k, & \text{if } k \text{ is a positive integer,} \\ 0, & \text{otherwise.} \end{cases}$$

   Now suppose we define a new random variable $Y$ as,

   $$Y = \begin{cases} 1, & \text{if } X \geq 10, \\ 0, & \text{if } X < 10. \end{cases}$$

   Find the PMF of $Y$.

   **Solution.**

   $$\begin{aligned} p_Y(0) &= P(\{X < 10\}) = P(\{X = 1\} \cup \{X = 2\} \cup \ldots \cup \{X = 9\}) \\ &= \sum_{i=1}^{9} P(\{X = i\}) = \sum_{i=1}^{9} \left(\frac{1}{2}\right)^i = 1 - \left(\frac{1}{2}\right)^9. \end{aligned}$$

   Since for $k$ other than 0,1, we have $p_Y(k) = 0$, it follows that $p_Y(1) = 1 - p_Y(0) = (1/2)^9$.

# MAT 271E – Homework 5

Due 10.04.2013

1. (a) Evaluate

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt.$$

(Hint : Compute $\int\int e^{-(t^2+u^2)}\,dt\,du$ using polar coordinates.)

(b) Using part (a), evaluate

$$\int_{-\infty}^{\infty} e^{-(t-\mu)^2/2\sigma^2}\,dt,$$

where $\mu$ and $\sigma$ are constants.

(c) Show that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\,t\,e^{-(t-\mu)^2/2\sigma^2}\,dt = \mu,$$

where $\sigma$ is a positive constant.
(Hint : First show that $\int t\,e^{-t^2}\,dt = 0$.)

(d) Show that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\,(t-\mu)^2\,e^{-(t-\mu)^2/(2\sigma^2)}\,dt = \sigma^2,$$

where $\mu$ is a constant.
(Hint : First show that $\int t^2\,e^{-t^2/2}\,dt = \sqrt{2\pi}$. You can make use of the polar coordinate trick in part (a) for this.)

**Solution.** (a) Let $c = \int_{-\infty}^{\infty} e^{-t^2}\,dt$. Observe that,

$$\begin{aligned} c^2 &= \int_{-\infty}^{\infty} e^{-t^2}\,dt \int_{-\infty}^{\infty} e^{-u^2}\,du \\ &= \iint_{-\infty}^{\infty} e^{-(t+u)^2}\,dt\,du. \end{aligned}$$

Now make a change of variables so as to write this in polar coordinates $(r,\theta)$, where,

$$t = r\,\cos(\theta), \quad u = r\,\sin(\theta),$$

so that $t^2+u^2 = r^2$. We have,

$$\begin{aligned} c^2 &= \int_0^{\infty}\int_0^{2\pi} r\,e^{-r^2}\,d\theta\,dr \\ &= 2\pi\int_0^{\infty} r\,e^{-r^2}\,dr \\ &= \pi. \end{aligned}$$

Therefore, $\int_{-\infty}^{\infty} e^{-t^2}\,dt. = \sqrt{\pi}$.

(b) Let $s = (t-\mu)/(\sqrt{2}\sigma)$. Then,

$$\begin{aligned} \sqrt{\pi} &= \int_{-\infty}^{\infty} e^{-s^2}\,ds \\ &= \int_{-\infty}^{\infty} e^{-(t-\mu)^2/2\sigma^2}\,\frac{1}{\sqrt{2}\,\sigma}dt. \end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-(t-\mu)^2/2\sigma^2}\,dt = \sqrt{2\pi}\,\sigma.$$

(c) Observe that $f(s) = s\,e^{-s^2/2\sigma^2}$ is an odd function (i.e. $f(-s) = -f(s)$). Therefore, $\int_{-\infty}^{\infty} f(s)\,ds = 0$. Now, let $s = t-\mu$. Then,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\,t\,e^{-(t-\mu)^2/2\sigma^2}\,dt &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\,(s+\mu)\,e^{-s^2/2\sigma^2}\,ds \\ &= \frac{1}{\sqrt{2\pi}\,\sigma}\left[\int_{-\infty}^{\infty} s\,e^{-s^2/2\sigma^2}\,ds + \int_{-\infty}^{\infty} \mu\,e^{-s^2/2\sigma^2}\,ds\right] \\ &= \mu\end{aligned}$$

(d) We follow the hint but do not make use of polar coordinates. Let $f(t) = -t\,e^{-t^2/2}$. Then, $f'(t) = t^2\,e^{-t^2/2} - e^{-t^2/2}$. From this observation, it follows that,

$$\begin{aligned}0 &= \int_{-\infty}^{\infty} f'(t)\,dt \\ &= \int_{-\infty}^{\infty} t^2\,e^{-t^2/2}\,dt - \int_{-\infty}^{\infty} e^{-t^2/2}\,dt\end{aligned}$$

By making a change of variables, from part (a), we deduce therefore that $\int t^2\,e^{-t^2/2}\,dt = \sqrt{2\pi}$. Now let $t = (s-\mu)/\sqrt{\sigma}$. We have,

$$\begin{aligned}\sqrt{2\,\pi} &= \int_{-\infty}^{\infty} t^2\,e^{-t^2/2}\,dt \\ &= \frac{1}{\sigma^3}\int_{-\infty}^{\infty} (s-\mu)^2\,e^{-(s-\mu)^2/(2\sigma^2)}\,ds\end{aligned}$$

Thus, it follows that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\,(s-\mu)^2\,e^{-(s-\mu)^2/(2\sigma^2)}\,ds = \sigma^2.$$

2. Suppose that $X_1$, $X_2$, ..., $X_n$ is a collection of independent random variables (i.e. the events defined in terms of different $X_i$'s are independent) associated with some experiment. Suppose we perform the experiment and order the values of these random variables in ascending order. The value that lies in the middle is called the 'median' of the sample (i.e. if after the experiment, for $n = 3$, it turns out that $X_1 = 2$, $X_2 = 4.5$, $X_3 = 2.1$, then the median is 2.1).

   (a) Let $n = 3$ and $Z$ be the median of $X_1$, $X_2$, $X_3$. Suppose that each $X_i$ is uniformly distributed on $[0, 1]$. Find the pdf of $Z$.
   (Hint : Think of the event $\{Z < t\}$ and express it as a union of non-intersecting events in terms of $X_i$'s.)

   (b) Repeat (a) for $n = 5$.

**Solution.** (a) We can write the event $E = \{Z < t\}$ as,

$$\begin{aligned}E = &\underbrace{\Big[\{X_1 < t\}\cap\{X_2 < t\}\cap\{X_3 \geq t\}\Big]}_{E_1} \cup \underbrace{\Big[\{X_1 < t\}\cap\{X_2 \geq t\}\cap\{X_3 < t\}\Big]}_{E_2} \\ &\cup \underbrace{\Big[\{X_1 \geq t\}\cap\{X_2 < t\}\cap\{X_3 < t\}\Big]}_{E_3} \cup \underbrace{\Big[\{X_1 < t\}\cap\{X_2 < t\}\cap\{X_3 < t\}\Big]}_{D}.\end{aligned}$$

Oberve that the events $E_1$, $E_2$, $E_3$, $D$ are distinct (i.e. the intersection of any two is $\emptyset$). Therefore,

$$P(E) = P(E_1) + P(E_2) + P(E_3) + P(D).$$

Now, since the random variables $X_1$, $X_2$, $X_3$ are independent, we have,

$$\begin{aligned} P(E_1) &= P(\{X_1 < t\}) \cdot P(\{X_2 < t\}) \cdot P(\{X_3 \geq t\}) \\ &= t \cdot t \cdot (1 - t) \\ &= t^2 - t^3 \end{aligned}$$

where $t \in [0, 1]$. We similarly have that $P(E_2) = P(E_3) = t^2 - t^3$. Also, we compute $P(D) = t^3$. Thus, $P(E) = 3t^2 - 2\,t^3$. Differentiating, we find

$$f_Z(t) = \begin{cases} 6t - 6t^2, & \text{for } 0 \leq t \leq 1, \\ 0, & \text{for } t \notin [0, 1]. \end{cases}$$

(b) Again, we will first compute the probability of the event $E = \{Z < t\}$, for $t \in [0, 1]$ – this will give us the CDF and we will differentiate the CDF to obtain the PDF. Consider now the events,

$$C_k = \{k \text{ of the } X_i\text{'s are strictly less than } t,\ n{-}k \text{ of the } X_i\text{'s are greater than or equal to } t\}.$$

Observe that $C_k$'s are disjoint, i.e. $C_k \cap C_j = \emptyset$ if $j \neq k$ (why?). Observe also that (again : why?),

$$E = C_3 \cup C_4 \cup C_5,$$

so that $P(E) = P(C_3) + P(C_4) + P(C_5)$. (Question : How does this generalize the solution given for part (a)?). We now compute,

$$\begin{aligned} P(C_3) &= \binom{5}{3}\, t^3\,(1 - t)^2 = 10\, t^3\,(1 - t)^2, \\ P(C_4) &= \binom{5}{4}\, t^4\,(1 - t) = 5\, t^4\,(1 - t), \\ P(C_4) &= \binom{5}{5}\, t^5 = t^5. \end{aligned}$$

Now differentiating $P(E)$ with respect to $t$ (do it – you will see that there are some cancellations), we obtain

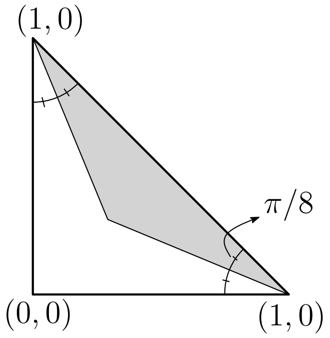
$$f_Z(t) = \begin{cases} 30\, t^2\,(1 - t)^2, & \text{if } t \in [0, 1], \\ 0, & \text{if } t \notin [0, 1]. \end{cases}$$

# MAT 271E – Homework 6

Due 17.04.2013

1. Consider the right triangle formed by the points $(0,0)$, $(0,1)$, $(1,0)$. Suppose we randomly pick a point inside this triangle. What is the probability that the point is closer to the hypotenuse than the other edges.

   **Solution.** Notice that this event is equivalent to the event that the selected point lies in the shaded area.

   Since any point in the larger triangle is equally likely, the probability of the event of interest is given by

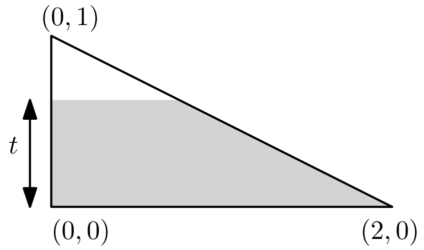
   $$\frac{\text{Shaded Area}}{\text{Area of the larger triangle}} = \frac{\sin(\pi/8)}{1/2}.$$

2. Consider the triangle formed by the points $(0,0)$, $(0,1)$, $(2,0)$. Suppose we randomly pick a point inside this triangle. Let $X$ denote the height of the point (the second coordinate).

   (a) Find $f_X(t)$, the pdf of $X$.

   (b) Find the expected height of the point.

   **Solution.** (a) Consider the event $A = \{X \leq t\}$. Observe that this event is equivalent to the event that the randomly picked point lies in the shaded trapezoid.

   Since any point in the triangle is equally likely (i.e. we have a uniform distribution), the probability of $A$ is given by

   $$P(A) = \frac{\text{Area of the trapezoid}}{\text{Area of the triangle}} = 2t - t^2.$$

   Thus the cdf of $X$ is given by,

   $$F_X(t) = \begin{cases} 1 & \text{if } 1 \leq t, \\ 2t - t^2 & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t < 0. \end{cases}$$

   Differentiating $F_X(t)$, we find the pdf as,

   $$f_X(t) = \begin{cases} 0 & \text{if } 1 \leq t, \\ 2 - 2t & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t < 0. \end{cases}$$

(b) Using the pdf found in part (a),

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} t\, f_X(t)\, dt = \int_0^1 2t - 2t^2\, dt = 1/3.$$

# MAT 271E – Homework 7

Due 24.04.2013

1. Suppose we break a stick of unit length into two pieces randomly.

   (a) Let $X$ be the length of the longer piece. Find the pdf of $X$. What is the expected value of $X$?

   (b) Suppose now that we break the longer piece into two pieces randomly. Let $Z$ be the longer of the resulting two pieces. Find the pdf of $Z$. What is the expected value of $Z$?

**Solution.** (a) Let $X_1$ denote the length of the piece on the left. Observe that $X_1$ is uniformly distributed on $[0, 1]$. Then, we will have, $X = \max\big(X_1, 1 - X_1\big)$. Consider the event $A = \{X \leq t\}$. Observe that, for $t \geq 1/2$,

$$\{X \leq t\} = \{X_1 \leq t\} \cup \{1 - X_1 \leq t\} = \{1 - t \leq X_1 \leq t\}.$$

Therefore, the cdf of $X$ satisfies,

$$F_X(t) = P(A) = 2t - 1,$$

for $1/2 \leq t \leq 1$. Noting also that $P(\{X \leq 1/2\}) = 0$ and $P(\{X \leq 1\}) = 1$ (why?), we obtain,

$$F_X(t) = \begin{cases} 1, & \text{if } 1 < t, \\ 2t - 1, & \text{if } 1/2 \leq t \leq 1, \\ 0, & \text{if } t < 1/2. \end{cases}$$

Differentiating, we obtain the pdf of $X$ as,

$$f_X(t) = \begin{cases} 2, & \text{if } 1/2 \leq t \leq 1, \\ 0, & \text{if } t \notin [1/2, 1]. \end{cases}$$

Using the pdf, we compute the expected value of $X$ as,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} t\, f_X(t)\, dt = \int_{1/2}^{1} t\, 2\, dt = \frac{3}{4}.$$

(b) Consider the event $B = \{Z \leq s\}$. To compute the probability of $B$, we will condition on the event $C = \{X = t\}$. Recall that in this case,

$$P(B) = \int_{-\infty}^{\infty} P\big(B|\{X = t\}\big)\, f_X(t)\, dt.$$

Now, suppose that the length of the longer piece in the first break is $X = t$. We break this stick of length $t$ into two randomly. Let $Z_1$ denote the length of the piece on the left. Then, we will have that $Z = \max\big(Z_1, t - Z_1\big)$. Observe that $Z_1$ is uniformly distributed on $[0, t]$. In this setting (i.e. under $\{X = t\}$), by reasoning as in part (a), we find conditional probability of $B$ as

$$\begin{aligned} P\big(\{Z \leq s\}|\{X = t\}\big) &= P\big(\{Z_1 \leq s\} \cup \{t - Z_1 \leq s\}|\{X = t\}\big) \\ &= P\big(\{t - s \leq Z_1 \leq s\}|\{X = t\}\big) \\ &= (2s - t)\, \frac{1}{t}. \end{aligned}$$

for $t/2 \leq s \leq t$. Thus,

$$\begin{aligned} P\big(\{Z \leq s\}\big) &= \int_{-\infty}^{\infty} P\big(\{Z \leq s\}|\{X = t\}\big)\, f_X(t)\, dt \\ &= \int_{1/2}^{1} \frac{2s - t}{t}\, 2\, dt \\ &= \end{aligned}$$

2. Suppose that $g(\cdot)$ is a monotone increasing function (i.e. $g(t_1) < g(t_2)$ if $t_1 < t_2$). Also let $h\big(g(t)\big) = t$. Finally, let $Y = g(X)$, where $X$ is a random variable with pdf $f_X(t)$. Show that

$$f_Y(t) = f_X\big(h(t)\big)\, h'(t).$$

3. Suppose that $X$ is a Gaussian random variable with mean $\mu$ and variance $\sigma^2$. Let $Y = a\,X + b$. Show that $Y$ is also a Gaussian random variable. Find the mean and the variance of $Y$.

MAT 271E – Probability and Statistics

Midterm Examination I

20.03.2013

5 Questions, 120 Minutes

(20 pts) 1. You have a regular, unbiased coin. You start tossing the coin. You stop if

- you observe two Heads before the fourth toss,

  or,

- you tossed the coin four times.

(a) Propose a sample space for this experiment.

(b) For the experiment described above, let us define the events

$$B_k = \{\text{a Head occurs at the } k^{\text{th}} \text{ toss}\}, \text{ for } k = 1, 2, 3, 4,$$
$$A = \{\text{you toss the coin three times and stop}\}.$$

Express the event $A$ in terms of the events $B_k$, for $k = 1, 2, 3, 4$.
(You can use as many $B_k$'s as you like).

(20 pts) 2. You roll a fair die 10 times. Assume that the rolls are independent. Compute the probability that you roll a '6' at least twice.

(20 pts) 3. Suppose your friend rolls a fair die twice. Assume that the rolls are independent. Your friend tells you that the first roll is not a 4. Answer the following questions by taking into account this information.

(a) Find the probability that the first roll is equal to 5.

(b) Find the probability that the second roll is greater than the first roll.

(20 pts) 4. Let $X$ and $Y$ be random variables whose joint PMF is given by the table below.

| | | $X$ | | | |
|---|---|---|---|---|---|
| | | -1 | 0 | 1 | 2 |
| | -1 | 2/16 | 1/16 | 1/16 | 0 |
| $Y$ | 0 | 0 | 1/16 | 0 | 2/16 |
| | 1 | 1/16 | 3/16 | 0 | 1/16 |
| | 2 | 3/16 | 0 | 1/16 | 0 |

(a) Find the expected value of $Y$.

(b) Compute the probability of the event $\{X \geq |Y|\}$

(20 pts) 5. There are 10 balls in an urn, numbered from 1 to 10. You draw a ball and then without putting it back, draw another ball. Let $X$ be the number on the first ball. Let $Y$ be the number on the second ball.

(a) Find $P_X(k)$, the probability mass function (PMF) of $X$.

(b) Find $P_Y(k)$, the PMF of $Y$.

(c) Let us define a new random variable $Z = X + Y$. Find the expected value of $Z$.

(Hint : Recall that $\sum_{n=1}^{k} = \frac{k\,(k+1)}{2}$.)

MAT 271E – Probability and Statistics

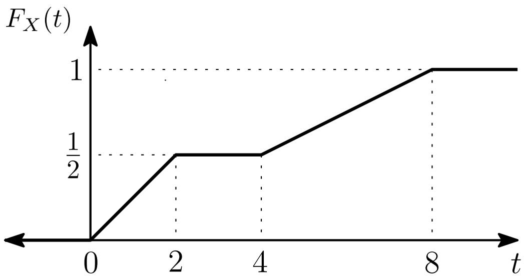
Midterm Examination II

24.04.2013

4 Questions, 100 Minutes

(25 pts) 1. Let $X$ be a continuous random variable whose cumulative distribution function (cdf) is given as,

$$F_X(t) = \begin{cases} 0, & \text{if } t < 0, \\ t/4, & \text{if } 0 \leq t \leq 2, \\ 1/2, & \text{if } 2 < t \leq 4, \\ t/8, & \text{if } 4 < t \leq 8, \\ 1, & \text{if } 8 < t, \end{cases}$$




as shown on the right.

(a) Find the probability of the event $A = \{X \geq 1\}$.

(b) Given the event $A = \{X \geq 1\}$, find the conditional probability of $B = \{2 \leq X \leq 5\}$.

(c) Given the event $A = \{X \geq 1\}$, determine and sketch $f_{X|A}(t)$, the conditional probability density function of $X$.

(25 pts) 2. Let $X$ be a random variable, uniformly distributed on $[0, 2]$. Also, let $Y = \lfloor X^2 \rfloor$, where $\lfloor \cdot \rfloor$ is the 'floor function' defined as,

$\lfloor t \rfloor$ = Greatest integer less than or equal to $t$.

(For example $\lfloor 1.2 \rfloor = 1$, $\lfloor 3.99 \rfloor = 3$.)

(a) Find the probability of the event $\{Y = 2\}$.

(b) Given the event $B = \{X \leq 3/2\}$, compute the conditional probability of the event $\{Y = 2\}$.

(25 pts) 3. Let $X$ and $Y$ be independent random variables. Assume that both $X$ and $Y$ are uniformly distributed on $[0, 1]$.

(a) Compute the probability of the event $A = \{X \leq Y\}$.

(b) Given the event $A = \{X \leq Y\}$, compute the probability that '$X \leq 1/2$'.

(c) Given the event $A = \{X \leq Y\}$, determine $F_{X|A}(t)$, the conditional cumulative distribution function (cdf) of $X$.

(d) Given the event $A = \{X \leq Y\}$, find the conditional expectation of $X$.

(25 pts) 4. Let $X$ be a random variable, uniformly distributed on $[-1, 1]$. Also, let $Y = \max(X, X^2)$.

(a) Compute the probability of the event $A = \{Y \leq 1/2\}$.

(b) Find the cumulative distribution function (cdf) of $Y$.

(c) Find $\mathbb{E}(Y)$, the expected value of $Y$.

MAT 271E – Probability and Statistics
(CRN : 21239, Instructor : İlker Bayram)

Final Examination

28.05.2013

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work for Full Credit!

(20 pts) 1. An urn initially contains 5 white balls and 1 black ball. Your friend removes 2 balls from the urn randomly so that there are now a total of 4 balls.

(a) What is the probability that the black ball is still in the urn?

(b) Suppose you pick randomly one of the remaining balls in the urn. What is the probability that the ball you pick is black?

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(20 pts) 2. You are given a biased coin for which the probability of observing a Head is 1/4. You toss the coin 10 times. Assume that the tosses are independent. Given this experiment, let us define the events

$A_k = \{\underline{\text{at least}}\ k \text{ Heads occur}\}$

for $k \geq 1$.

(a) Compute the probability of the event $A_1$.

(b) Given that the event $A_1$ has occured, compute the probability of $A_2$. That is, compute $P(A_2|A_1)$.

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(20 pts) 3. Let $X$ and $Y$ be independent random variables that have the same probability density function (pdf) given as,

$$f_X(t) = f_Y(t) = \begin{cases} 2t, & \text{for } 0 \leq t \leq 1, \\ 0, & \text{for } t \notin [0,1]. \end{cases}$$

Also, let $Z$ be a random variable defined as $Z = (Y - X)^2$.

(a) Compute the mean and the second moment of $X$, that is $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$.

(b) Compute the mean of $Z$, that is $\mathbb{E}(Z)$.

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(20 pts) 4. Let $X$ and $Y$ be independent random variables. Suppose that $X$ is uniformly distributed on $[0,1]$. Also, let the probability density function (pdf) of $Y$ be given as,

$$f_Y(t) = \begin{cases} e^{-t}, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

(a) Compute the probability of the event $\{s \leq Y\}$, where $s \geq 0$.

(b) Compute the probability of the event $\{X \leq Y\}$.

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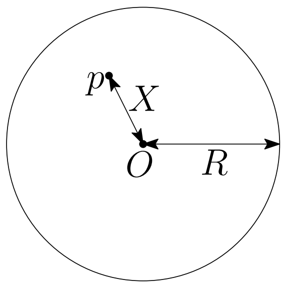
(20 pts) 5. Consider a disk with unknown radius $R$, centered around the origin $O$. Suppose your friend picks a point $p$ on this disk randomly (i.e., any point is equally likely). Also, let the distance of $p$ to the origin be denoted as $X$ (see the figure).




(a) Find the probability of the event $\{X \leq t\}$ (in terms of $R$) for $0 \leq t \leq R$.

(b) Find $f_X(t)$, the pdf of $X$.

(c) Consider $\hat{R} = 3X/2$ as an estimator for $R$. Is $\hat{R}$ biased or unbiased? If it is biased, propose an unbiased estimator for $R$. If it is unbiased, explain why.

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