MAT 271E – Probability and Statistics

Spring 2014

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Class Meets : 13.30 – 16.30, Wednesday
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Textbook : D. B. Bertsekas, J. N. Tsitsiklis, 'Introduction to Probability', 2^{nd} Edition, Athena-Scientific.

Grading : Homeworks (10%), 2 Midterms (25% each), Final (40%).

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Tentative Course Outline

- Probability Space
  Probability models, conditioning, Bayes' rule, independence.
- Discrete Random Variables
  Probability mass function, functions of random variables, expectation, joint PMFs, conditioning, independence.
- General Random Variables
  Probability distribution function, cumulative distribution function, continuous Bayes' rule, correlation, conditional expectation.
- Limit Theorems
  Law of large numbers, central limit theorem.
- Introduction to Statistics
  Parameter estimation, linear regression, hypothesis testing.

MAT 271E – Homework 1

Due 26.02.2014

1. Suppose that a die is loaded so that the probabilities of different outcomes are different. However, we learn somehow that,

$$P(\{\text{outcome is even}\}) = 2/3,$$
$$P(\{\text{outcome } > 1\}) = 3/4.$$

Find the probability that the outcome is either 3 or 5.

Solution. Let us denote the events as $A = \{\text{outcome is even}\}$, $B = \{\text{outcome } > 1\}$, $C = \{\text{outcome is either 3 or 5}\})$. Then, C and B^c are disjoint (i.e. $B^c \cap C = \emptyset$) and $B^c \cup C = A^c$. Therefore, $P(A^c) = P(B^c) + P(C)$ by the additivity of the probability law. But $P(A^c) = 1 - P(A) = 1/3$ and $P(B^c) = 1 - P(B) = 1/4$. Thus, $P(C) = 1/3 - 1/4 = 1/12$. $\square$

2. Suppose that A and B are two events and $P(A) = 1/2$, $P(B^c) = 1/3$. Can A and B be disjoint? If so, describe a setting such that this is the case. If not, explain why not.

Solution. From the given information, we find that $P(B) = 1 - P(B^c) = 2/3$. If A and B were disjoint, then we would have, by the additivity property of the probability law that,

$$P(A \cup B) = P(A) + P(B) = 1/2 + 2/3 > 1,$$

which is impossible. Therefore, the two events cannot be disjoint. $\square$

3. We roll a fair die 5 times.

(a) What is the probability that we observe a '6' at least twice?

(b) What is the probability that we observe a '6' at least twice, given that a '6' is observed at least once?

Solution. Let the events A_1 and A_2 be defined as,

$$A_1 = \{\text{we observe a '6' at least once}\},$$
$$A_2 = \{\text{we observe a '6' at least twice}\}.$$

(a) We need to compute $P(A_2)$. But $P(A_2^c)$ is easier to compute. Observe that A_2^c can be expressed as the union of two disjoint events B_0, B_1, where,

$$B_0 = \{\text{we do not observe a '6'}\},$$
$$B_1 = \{\text{we observe a '6' only once}\}.$$

Now, if we take the rolls as independent, we find the probability of not observing a '6' in five rolls (which is B_0) as,

$$P(B_0) = \left(\frac{5}{6}\right)^5.$$

To compute $P(B_1)$ it would be easier to use tricks from combinatorics for this case, but we can also compute $P(B_1)$ without resorting to combinatorics – we will instead make use of conditional probabilities. For that, let us define the events,

$$C_1 = \{\text{we observe a single '6' in the first roll}\},$$
$$C_2 = \{\text{we observe a single '6' in the first two rolls}\},$$
$$\vdots$$
$$C_k = \{\text{we observe a single '6' in the first } k \text{ rolls}\},$$

Note that C_5 is equivalent to B_1. So, we need to compute $P(C_5)$. Now observe that, we can write, by the total probability theorem, for $k > 1$,

$$P(C_k) = P(C_k|C_{k-1})\, P(C_{k-1}) + P(C_k|C_{k-1}^c)\, P(C_{k-1}^c).$$

Note that $P(C_k|C_{k-1}) = 5/6$ and $P(C_k|C_{k-1}^c) = 1/6$. Let us also denote $p_k = P(C_k)$ for the simplicity of notation. Then, we have,

$$\begin{aligned} p_k &= \frac{5}{6}\, p_{k-1} + \frac{1}{6}\, (1 - p_{k-1}) \\ &= \frac{1}{6} + \frac{4}{6}\, p_{k-1}. \end{aligned}$$

Thus, we can write, by iterating this equality, for $k > n$,

$$\begin{aligned} p_k &= \frac{1}{6} + \frac{4}{6}\, p_{k-1} \\ &= \frac{1}{6} + \frac{4}{6}\left(\frac{1}{6} + \frac{4}{6}\, p_{k-2}\right) = \frac{1}{6} + \frac{4}{6}\,\frac{1}{6} + \left(\frac{4}{6}\right)^2 p_{k-2} \\ &\;\vdots \\ &= \frac{1}{6} + \frac{4}{6}\,\frac{1}{6} + \left(\frac{4}{6}\right)^2 \frac{1}{6} + \ldots + \left(\frac{4}{6}\right)^{n-1} \frac{1}{6} + \left(\frac{4}{6}\right)^n p_{k-n} \end{aligned}$$

Note also that $p_1 = 1/6$. Setting $k = 5$, and $n = 4$, we thus obtain,

$$p_5 = \frac{1}{6}\sum_{i=0}^{4}\left(\frac{4}{6}\right)^i = \frac{1}{6}\,\frac{1-(4/6)^5}{1-(4/6)} = \frac{1-(4/6)^5}{2}.$$

We can now compute $P(A_2) = 1 - P(B_0) - p_5$.

(b) Before we begin, observe that if A_1 does not occur (we do note observe any '6's), then A_2 cannot occur (we cannot observe a '6' at least twice). This implies that $A_2 \subset A_1$. Therefore, $A_2 \cap A_1 = A_2$.

In the following, we will need $P(A_1)$. But $P(A_1^c)$ is easier to compute. Observe that A_1^c is equivalent to B_0 above. Therefore,

$$P(A_1) = 1 - \left(\frac{5}{6}\right)^5.$$

Now, the conditional probability is computed as (all of the terms are computed above),

$$P(A_2|A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{P(A_2)}{P(A_1)}.$$

MAT 271E – Homework 2

Due 12.03.2014

1. You roll a fair die until you observe a '1'. Let X be the number of '6's that you observe.

   (a) Suppose we define the event $A_n = \{\text{you roll for } n \text{ times}\}$. Find the probability of A_n.

   (b) Find the probability mass function (PMF) of X.

   (Hint : For parts (b) try conditioning on A_n.)

   Solution. (a) Note that A_n occurs if the first $n-1$ rolls is not 1 and the n^{th} roll is 1. Since the rolls are independent, we thus find

   $$P(A_n) = \left(\frac{5}{6}\right)^{n-1} \frac{1}{6}.$$

   (b) Let us define the event

   $$B_k = \{\text{we observe '}k\text{' 6's}\}.$$

   Note that B_k is equivalent to the event $\{X = k\}$. Its probability is easier to compute if we condition on A_n. Specifically, if $k \leq n$, then (why?)

   $$P(B_k|A_{n+1}) = \binom{n}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{n-k}.$$

   Note at this point that $A_n \cap A_k = \emptyset$ if $n \neq k$ and

   $$\bigcup_{n=1}^{\infty} A_n = \Omega.$$

   Thus A_n's partition Ω.
   Since A_n's partition Ω, we have

   $$\begin{aligned} P_X(k) &= P(B_k) \\ &= \sum_{n=k}^{\infty} P(B_k|A_{n+1})\, P(A_{n+1}) \\ &= \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{n-k} \left(\frac{5}{6}\right)^n \frac{1}{6} \\ &= \frac{1}{6}\left(\frac{1}{4}\right)^k \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{2}{3}\right)^n. \end{aligned}$$

2. Let X be a geometric random variable with parameter p. Recall that the PMF of X is given as

   $$P_X(k) = \begin{cases} p\,(1-p)^{k-1}, & \text{if } k \text{ is a positive integer,} \\ 0, & \text{otherwise.} \end{cases}$$

   Also, let Y be another random variable defined as $Y = X - 1$. Express the PMF of Y, namely $P_Y(\cdot)$, in terms of $P_X(\cdot)$.

   Solution. Observe that for $k \geq 1$, the equivalence of the events $\{X = k\} = \{Y + 1 = k\} = \{Y = k - 1\}$ imply that

   $$\begin{aligned} P_X(k) &= P(\{X = k\}) \\ &= P(\{Y = k-1\}) \\ &= P_Y(k-1) \end{aligned}$$

   for $k \geq 1$. Changing variables (as $n = k - 1$), we obtain,

   $$\begin{aligned} P_Y(n) &= P_X(n+1) \quad \text{for } n \geq 0 \\ &= p\,(1-p)^n, \quad \text{for } n \geq 0. \end{aligned}$$

3. Let X be a random variable that takes integer values. Also, let $P_X(\cdot)$ denote its PMF.

   (a) Suppose we define $Y = |X|$. Express $P_Y(\cdot)$ in terms of $P_X(\cdot)$.

   (b) Suppose we define $Z = X^2$. Express $P_Z(\cdot)$ in terms of $P_X(\cdot)$.

   Solution. (a) Notice that we can write the event $\{Y = k\}$ as,

$$\{Y = k\} = \begin{cases} \{X = k\} \cup \{X = -k\}, & \text{if } k > 0, k \in \mathbb{Z}, \\ \{X = 0\}, & \text{if } k = 0, \\ \emptyset, & \text{if } k < 0. \end{cases}$$

   Thus,

$$P_Y(k) = P\{Y = k\} = \begin{cases} P\{X = k\} + P\{X = -k\} = P_X(k) + P_X(-k), & \text{if } k > 0, k \in \mathbb{Z} \\ P\{X = 0\} = P_X(0), & \text{if } k = 0, \\ P(\emptyset) = 0, & \text{if } k < 0. \end{cases}$$

   (b) Let now A be the set of integers whose square roots are integers.

$$\{Y = k\} = \begin{cases} \{X = \sqrt{k}\} \cup \{X = -\sqrt{k}\}, & \text{if } k \in A, \\ \{X = 0\}, & \text{if } k = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

   Thus, we obtain,

$$P_Y(k) = \begin{cases} P_X(\sqrt{k}) + P_X(-\sqrt{k}), & \text{if } k \in A, \\ P_X(0), & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

4. Let X be a geometric random variable with parameter p, whose PMF is as given in Question-2 above. Also, let Z be a random variable defined as $Z = 1/X$. Find $\mathbb{E}(Z)$, the expected value of Z.

   Solution. Note that the expected value of Z is

$$\mathbb{E}(Z) = \sum_{k=1}^{\infty} \frac{1}{k}\, p\,(1-p)^{k-1} = \frac{p}{1-p} \underbrace{\sum_{k=1}^{\infty} \frac{1}{k}\,(1-p)^k}_{h(p)}. \tag{1}$$

   We need to evaluate $h(p)$. Notice that for $0 < p < 1$,

$$\begin{aligned} h'(p) &= \sum_{k=1} (1-p)^{k-1} \\ &= \sum_{k=0} (1-p)^k \\ &= \frac{1}{1-(1-p)} = \frac{1}{p}. \end{aligned}$$

   Thus, $h(p) = \ln(p) + c$ for some constant c. To find c, observe from (1) that

$$\lim_{p\to 1} h(p) = 0.$$

   Since $\lim_{p\to 1} \ln(p) = 0$, we must have $c = 0$. Thus, $h(p) = \ln(p)$. Plugging this in (1), we find that $\mathbb{E}(Z) = \frac{p}{1-p} \ln(p)$.

5. In this question, we find a simple expression for the mean of a binomial random variable. Recall that if X is a binomial random variable with parameters $(1-p)$, n, then the PMF of X is

$$P_X(k) = \begin{cases} \binom{n}{k} (1-p)^k\, p^{n-k} & \text{if } k \in \{0, 1, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Suppose we define a function of p as,

$$f(p) = \sum_{k=1}^{n} \binom{n}{k} \left(\frac{1-p}{p}\right)^{k}.$$

Find a simple expression for $f(p)$.

(b) Differentiate $f(p)$ – carry out differentiation for both the definition that involves the sum above and the simple expression you found in part (a), so as to obtain an equality. Then rearrange the equality so as to obtain a simple expression for

$$\mathbb{E}(X) = \sum_{k=0}^{n} k \binom{n}{k} (1-p)^{k} p^{n-k}.$$

Solution. (a) Using the binomial theorem we can write,

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} (1-p)^{k} p^{n-k} &= p^{n} + p^{n} \underbrace{\sum_{k=0}^{n} \binom{n}{k} (1-p)^{k} p^{-k}}_{f(p} \\ &= 1.\end{aligned}$$

From this, we obtain $f(p) = p^{-n} - 1$.

(b) If we differentiate both of the the expressions for $f(p)$, we obtain an equality as,

$$\sum_{k=1}^{n} \binom{n}{k} k \left(\frac{1-p}{p}\right)^{k-1} \left(-\frac{1}{p^2}\right) = -\sum_{k=1}^{n} \binom{n}{k} k\,(1-p)^{k-1} p^{-k-1} = -n\,p^{-n-1}.$$

Multiplying both sides of this equation by $-(1-p)\,p^{n+1}$, we obtain,

$$\sum_{k=1}^{n} k \binom{n}{k} (1-p)^{k} p^{n-k} = n\,(1-p).$$

Thus, $\mathbb{E}(X) = n\,(1-p)$.

MAT 271E – Homework 3

Due 02.04.2014

1. Let X_1, X_2 be independent discrete random variables with the same PMF, given as,

$$P_1(n) = P_2(n) = \begin{cases} (1-p)p^n & \text{if } n \text{ is a non-negative integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Now let $Y = X_1 + X_2$.

(a) Find the probability of the event $\{Y = 2\}$.

(b) Find the PMF of Y.

Solution. (a) We can express the event $\{Y = 2\}$ as,

$$\{Y = 2\} = \underbrace{\{X_1 = 0\} \cap \{X_2 = 2\}}_{E_0} \cup \underbrace{\{X_1 = 1\} \cap \{X_2 = 1\}}_{E_1} \cup \underbrace{\{X_1 = 2\} \cap \{X_2 = 0\}}_{E_2},$$

where E_i's are disjoint. Since X_1 and X_2 are disjoint, we have,

$$P\big(E_i\big) = P_X(i)\, P_X(2-i),$$

where we take $P_X = P_1 (= P_2)$. Since E_i's are disjoint, we have,

$$\begin{aligned} P\big(\{Y = 2\}\big) &= \sum_{i=0}^{2} P_X(E_i) \\ &= \sum_{i=0}^{2} (1-p)p^i\,(1-p)\,p^{2-i} \\ &= 3\,(1-p)^2\,p^2. \end{aligned}$$

(b) We need to evaluate the probability of the event $S_n = \{Y = n\}$ for non-negative n. Observe that as in (a) above, we can write

$$\{Y = n\} = \underbrace{\{X_1 = 0\} \cap \{X_2 = n\}}_{E_0} \cup \underbrace{\{X_1 = 1\} \cap \{X_2 = n-1\}}_{E_1} \cup \cdots \cup \underbrace{\{X_1 = n\} \cap \{X_2 = 0\}}_{E_n}.$$

Arguing as in part (a) above, we obtain, for $n \geq 0$,

$$\begin{aligned} P_Y(n) &= P\big(\{Y = n\}\big) \\ &= \sum_{i=0}^{n} P(E_i) \\ &= \sum_{i=0}^{n} (1-p)p^i\,(1-p)\,p^{n-i} \\ &= (n+1)\,(1-p)^2\,p^n. \end{aligned}$$

2. You roll a fair die once. Then, if the outcome Z is equal to n, you roll the fair die n more times to obtain n new outcomes $X_1, \ldots, X_n$ (assume that these rolls are independent). Let X be the sum of the X_i's.

(a) Find the expected value of X.

(b) Let $Y = X + Z$. Find the expected value of Y.

Solution. (a) Let us defined the event $E_n = \{Z = n\}$. Note that $P(E_n) = \frac{1}{6}$. Since $\mathbb{E}(X_i) = 7/2$, we have,

$$\mathbb{E}(X|E_n) = n\,\frac{7}{2}.$$

Since E_n's partition the sample space,

$$\mathbb{E}(X) = \sum_{i=1}^{6} \mathbb{E}(X|E_n)\,P(E_n) = \frac{7}{12} \sum_{i=1}^{6} n = \frac{147}{12}.$$

(b) Noting that $\mathbb{E}(Z) = 7/2$, we obtain,

$$\mathbb{E}(Y) = \mathbb{E}(X) + \mathbb{E}(Z) = \frac{147}{12} + \frac{7}{2} = \frac{189}{12}.$$

3. X is said to be a Laplacian random variable if it has a pdf of the form

$$f_X(t) = \frac{1}{2}\, e^{-|t|}.$$

Let Y be another random variable defined as $Y = |X|$.

(a) Find the pdf of Y. (Hint : Find the cdf and differentiate.)

(b) Find the probability of the events $E_1 = \{X > -1\}$, $E_2 = \{Y \leq 1\}$.

(c) Find the expected values of X and Y.

(d) Find the variances of X and Y.

Solution. (a) Note that for $t \geq 0$,

$$\begin{aligned}
F_Y(t) &= P\big(Y \leq t\big) \\
&= P\big(-t \leq X \leq t\big) \\
&= \int_{-t}^{t} \frac{1}{2}\, e^{-|t|} dt \\
&= \int_{-t}^{0} \frac{1}{2}\, e^{t} dt + \int_{0}^{t} \frac{1}{2}\, e^{-t} dt \\
&= 1 - e^{-t}.
\end{aligned}$$

Note also that, $F_Y(t) = P(Y \leq t) = 0$ for $t < 0$. Differentiating $F_Y(t)$ we obtain,

$$f_Y(t) = \begin{cases} e^{-t}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

(b) First,

$$\begin{aligned}
P(X > -1) &= \int_{-1}^{\infty} f_X(t)\, dt \\
&= \int_{-1}^{0} \frac{1}{2}\, e^{t}\, dt + \int_{0}^{\infty} \frac{1}{2}\, e^{-t}\, dt \\
&= \frac{1}{2}\big(1 - e^{-1} + 1\big) = 1 - \frac{1}{2}\, e^{-1}.
\end{aligned}$$

Observe that due to the symmetry of the pdf of X, if we define $E_3 = \{X < 1\}$, we have $P(E_3) = P(E_1)$. Now note that $E_2^c = E_1^c \cup E_2^c$ and $E_1^c \cap E_2^c = \emptyset$. Therefore, $P(E_2^c) = 2(1 - P(E_1))$. It follows that,

$$P(E_2) = 1 - 2(1 - P(E_1)) = 2P(E_1) - 1 = 1 - e^{-1}.$$

(c) Since $f_X(t) = f_X(-t)$,

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} t\, f_X(t)\, dt \\
&= \int_{-\infty}^{0} t\, f_X(t)\, dt + \int_{0}^{t} t\, f_X(t)\, dt \\
&= \int_{0}^{\infty} (-t)\, f_X(-t)\, dt + \int_{0}^{t} t\, f_X(t)\, dt \\
&= -\int_{0}^{\infty} t\, f_X(t)\, dt + \int_{0}^{t} t\, f_X(t)\, dt \\
&= 0.
\end{aligned}$$

For Y, we integrate by parts,

$$
\begin{aligned}
\mathbb{E}(Y) &= \int_0^\infty \underbrace{t}_{f} \underbrace{e^{-t}}_{g'} \, dt \\
&= \underbrace{t}_{f} \underbrace{(-e^{-t})}_{g} \Big|_0^\infty - \int_0^\infty \underbrace{1}_{f'} \underbrace{(-e^{-t})}_{g} \, dt \\
&= 1.
\end{aligned}
$$

(d) Let us start with Y. Again, we integrate by parts,

$$
\begin{aligned}
\mathbb{E}(Y^2) &= \int_0^\infty \underbrace{t^2}_{f} \underbrace{e^{-t}}_{g'} \, dt \\
&= \underbrace{t^2}_{f} \underbrace{(-e^{-t})}_{g} \Big|_0^\infty - \int_0^\infty \underbrace{2t}_{f'} \underbrace{(-e^{-t})}_{g} \, dt \\
&= 2.
\end{aligned}
$$

We obtain,

$$\mathrm{var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 1.$$

Now note, that $Y^2 = X^2$. So, $\mathbb{E}(Y^2) = \mathbb{E}(X^2)$ and therefore,

$$\mathrm{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2. \mathcal{VE}$$

MAT 271E – Homework 4

Due 30.03.2014

1. We randomly and independently pick two points on a circle. Find the probability that the chord between the two points is longer than the radius of the circle.

   Solution. Suppose p_1 is the first point we picked. Then, considering it fixed, we randomly pick another point p_2. Consider the angle formed by p_1, the center c and p_2. If this angle is less than $pi/3$, then the chord between p_1 and p_2 will be shorter than the radius. Therefore, for fixed p_1, the probability that the chord is longer than the radius is

$$\frac{2\pi - 2\pi/3}{2\pi} = 2/3.$$

   Since p_1 is selected according to a uniform distribution on the circle, the probability that the chord is longer than the radius is 2/3 too.

2. Suppose X is a random variable, uniformly distributed on $[0, 1]$. Also, let Z be a new random variable defined as $Z = \max(X, 1 - X)$.

   (a) Compute $\mathbb{E}(Z)$, without finding the pdf of Z.

   (b) Find $f_Z(t)$, the pdf of Z.

   Solution. (a) Since Z is a function of X, we can compute its expected value as,

$$\begin{aligned}
\mathbb{E}(Z) &= \mathbb{E}\big(\max(X, 1-X)\big) \\
&= \int_{-\infty}^{\infty} \max(t, 1-t)\, f_X(t)\, dt \\
&= \int_{0}^{1} \max(t, 1-t)\, dt \\
&= \int_{0}^{1/2} (1-t)\, dt + \int_{1/2}^{1} t\, dt \\
&= -\frac{1}{2}(1-t)^2\Big|_0^{1/2} + \frac{1}{2}t^2\Big|_{1/2}^{1} \\
&= 3/4.
\end{aligned}$$

   (b) To find $f_Z(t)$ we will first compute its cdf and then differentiate. Observe that, we can express the event $\{Z \leq t\}$ as,

$$\{Z \leq t\} = \{\max(X, 1-X) \leq t\} = \{X \leq t\} \cap \{1 - X \leq t\} = \{1 - t \leq X \leq t\}.$$

   Therefore,

$$F_Z(t) = P\{Z \leq t\} = \begin{cases} 0, & \text{if } t < 1/2, \\ 2t - 1, & \text{if } 1/2 \leq t < 1, \\ 1, & \text{if } 1 \leq t. \end{cases}$$

   Differentiating this we obtain,

$$f_Z(t) = \begin{cases} 0, & \text{if } t < 1/2, \\ 2, & \text{if } 1/2 \leq t < 1, \\ 0, & \text{if } 1 \leq t. \end{cases}$$

3. Suppose X and Y are independent random variables, uniformly distributed on $[0, 1]$. Also, let $Z = X + Y$. Find $f_Z(t)$, the pdf of Z.

   Solution. Recall that since X and Y are independent, we need to convolve $f_X(t)$ and $f_Y(t)$. That is,

$$f_Z(t) = \int_{-\infty}^{\infty} f_X(t-u)\, f_Y(u)\, du.$$

Note that,

$$f_X(t) = f_Y(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } 0 \le t < 1, \\ 0, & \text{if } 1 \le t. \end{cases}$$

In the following, we need to pay attention to the limits of integration. The expression will depend on the value of t.

First observe that if $t < 0$ or $t > 2$, the integral evaluates to 0 (Why?).

If $0 < t < 1$, to make both $f_X(t-u)$ and $f_Y(u)$ non-zero, both $(t-u)$ and u have to lie in the interval $[0, 1]$, so,

$$f_Z(t) = \int_0^t 1\, du = t \text{ for } 0 < t < 1$$

For $1 \le t < 2$, to make $(t-u)$ and u to lie in the interval $[0, 1]$, we need to take $u \in [t-1, 1]$, so,

$$f_Z(t) = \int_{t-1}^1 1\, du = 2 - t \text{ for } 1 \le t < 2.$$

Summing up,

$$f_Z(t) = \begin{cases} 0, & \text{if } t < 0, \\ t, & \text{if } 0 \le t < 1, \\ 2-t, & \text{if } 1 \le t < 2, \\ 0, & \text{if } 2 \le t. \end{cases}$$

Sketch this function to see what it looks like!

4. Suppose X and Y are independent random variables, whose pdfs are given as,

$$f_X(t) = \begin{cases} \lambda_1\, e^{-\lambda_1 t}, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0, \end{cases}$$

$$f_Y(t) = \begin{cases} \lambda_2\, e^{-\lambda_2 t}, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0, \end{cases}$$

where λ_1 and λ_2 are constants. Also, let $Z = X + Y$. Find $f_Z(t)$, the pdf of Z.

Solution. Because X and Y are independent, we need to convolve the two pdfs. Now let $t > 0$ and pay attention to limits of integration.

$$\begin{aligned} f_Z(t) &= \int_{-\infty}^{\infty} f_X(t-u)\, f_Y(u)\, du \\ &= \int_0^t \lambda_1\, e^{-\lambda_1 (t-u)}\, \lambda_2\, e^{-\lambda_2 u}\, du \\ &= \lambda_1\, \lambda_2\, e^{-\lambda_1 t} \int_0^t e^{(\lambda_1 - \lambda_2) u}\, du \\ &= \frac{\lambda_1\, \lambda_2}{\lambda_1 - \lambda_2}\, e^{-\lambda_1 t} \left(e^{(\lambda_1-\lambda_2)u} \Big|_0^t \right) \\ &= \frac{\lambda_1\, \lambda_2}{\lambda_1 - \lambda_2}\, e^{-\lambda_1 t} \left(e^{(\lambda_1-\lambda_2)t} - 1 \right) \\ &= \frac{\lambda_1\, \lambda_2}{\lambda_1 - \lambda_2} \left(e^{-\lambda_2 t} - e^{-\lambda_1 t} \right). \end{aligned}$$

Observe also, that if $t < 0$, the $f_X(t-u)\, f_Y(u)$ is zero for all values of u. Therefore,

$$f_Z(t) = \begin{cases} 0, & \text{if } t < 0, \\ \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left(e^{-\lambda_2 t} - e^{-\lambda_1 t} \right), & \text{if } 0 \le t. \end{cases}$$

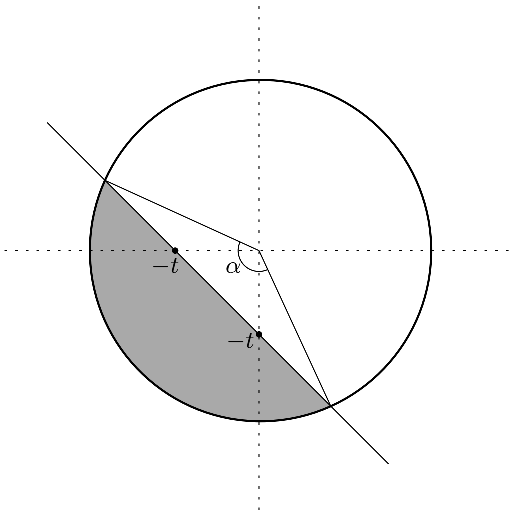
We know that pdfs must always be non-negative. Does the subtraction cause any trouble here?

5. Suppose X and Y are uniformly distributed on the unit disk. That is, their joint pdf is given as,

$$f_{X,Y}(t,u) = \begin{cases} \frac{1}{\pi}, & \text{if } t^2+u^2 \le 1, \\ 0, & \text{if } t^2+u^2 > 1. \end{cases}$$

Let $Z = X+Y$. Find $f_Z(t)$, the pdf of Z.

Solution. In this question, X and Y are not independent. We will find the pdf of Z by differentiating its cdf. Consider now the event $\{Z < -t\}$ for $0 < t < \sqrt{2}$. Note that this event occurs if in the figure below, the point chosen in the unit disk falls in the shaded area.



Since selecting any point in the disk is equally likely, we find the probability of this event by computing the ratio of the shaded area to the whole disk. Note that, α is related to t through,

$$\alpha = 2 \arccos\left(\frac{t}{\sqrt{2}}\right).$$

Using this, we find the area of the shaded region as,

$$\frac{\alpha}{2} - \frac{1}{2}\frac{t}{\sqrt{2}}\sqrt{4-2t^2} = \arccos\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2}\frac{t}{\sqrt{2}}\sqrt{4-2t^2}$$

Dividing by the total are of the disk, which is π, we find the mentioned probability. By symmetry, we write the cdf of Z as (check this!),

$$f_Z(t) = \begin{cases} 0, & \text{if } t < -\sqrt{2}, \\ \frac{1}{\pi}\left[\arccos\left(-\frac{t}{\sqrt{2}}\right) + \frac{1}{2}\frac{t}{\sqrt{2}}\sqrt{4-2t^2}\right], & \text{if } -\sqrt{2} \le t \le 0, \\ \frac{1}{\pi}\left[\left(\pi - \arccos\left(\frac{t}{\sqrt{2}}\right)\right) + \frac{1}{2}\frac{t}{\sqrt{2}}\sqrt{4-2t^2}\right], & \text{if } 0 \le t \le \sqrt{2}, \\ 1, & \text{if } \sqrt{2} \le t. \end{cases}$$

Differentiate to obtain the pdf!

6. Suppose X and Y are independent random variables, uniformly distributed on $[0,1]$. Also, let $Z = \min(X,Y)$. Find $f_Z(t)$, the pdf of Z.

Solution. We find the cdf and differentiate. Note that

$$\{Z > t\} = \{X > t\} \cap \{Y > t\}.$$

So, for $0 < t < 1$, we compute, using the independence of X and Y,

$$F_Z(t) = P\{Z \le t\} = 1 - P\{Z > t\} = 1 - P\{X > t\}\,P\{Y > t\} = 1 - (1-t)^2.$$

Complete the missing values of t to find,

$$F_Z(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1-(1-t)^2, & \text{if } 0 < t < 1, \\ 1, & \text{if } 1 \le t. \end{cases}$$

Differentiate to obtain,

$$f_Z(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 2(1-t), & \text{if } 0 < t < 1, \\ 0, & \text{if } 1 \leq t. \end{cases}$$

MAT 271E – Probability and Statistics

Midterm Examination I

19.03.2014

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work for Full Credit!

(20 pts) 1. An urn contains a red ball, a green ball, and a blue ball. A ball is drawn randomly from the urn. Then without putting the drawn ball back, another ball is drawn. Based on this experiment, two events, E_1 and E_2, are defined as,

E_1 : the first ball is red,

E_2 : the second ball is red.

(a) Propose a sample space for this experiment.

(b) Express the event '$E_1 \cup E_2$' in terms of the elements of the sample space.

(c) Express the event '$E_1 \cap E_2$' in terms of the elements of the sample space.

(20 pts) 2. Two specialists claim that they can tell whether a painting is original or a reproduction by looking at the painting. However it is known from past experience that each specialist makes mistakes with probability p. (Note that there are two types of mistakes – (i) the painting is original but the specialist decides it is a reproduction, (ii) the painting is a reproduction but the specialist decides it is original. The probability of both types of mistakes are the same and equal to p).

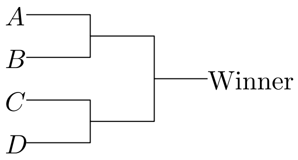
Suppose it is known that there are 3 reproductions and 7 originals in a collection of 10 paintings but this information is not provided to the specialists. A painting is selected randomly from the collection and is presented to the specialists. Assume that the specialists decide on their own (they do not know what the other decides).

(a) Find the probability that the first specialist decides that the given painting is original.

(b) Given that the first specialist decides the painting is original, find the conditional probability that the selected painting is actually a reproduction.

(c) Given that the first specialist decides the painting is original, find the conditional probability that the second painter decides the painting is original.

(20 pts) 3. There are four teams in a tournament, labelled A, B, C, D. Suppose the fixture is arranged as shown below.



We are given the following information regarding the matches between different teams.

$$P\{A \text{ beats } B\} = p_B,$$
$$P\{A \text{ beats } C\} = p_C,$$
$$P\{A \text{ beats } D\} = p_D,$$
$$P\{D \text{ beats } B\} = s_B,$$
$$P\{D \text{ beats } C\} = s_C.$$

Assume that the matches are independent.

(a) Find the probability that A wins the tournament.

(b) Find the probability that B or C wins the tournament.

(20 pts) 4. Suppose that the PMF of a random variable X is given as

$$P_X(k) = \begin{cases} \left(\frac{1}{3}\right)^{k-1} \frac{2}{3}, & \text{if } k \text{ is a positive integer,} \\ 0, & \text{otherwise.} \end{cases}$$

Also, let Y be another random variable defined as $Y = 2^X$.

(a) Find the PMF of Y.

(b) Find $\mathbb{E}(Y)$, the expected value of Y.
(Note : Recall that '$\sum_{n=1}^{\infty} p^n = p/(1-p)$' if $|p| < 1$.)

(20 pts) 5. A fair die is rolled once. Based on this experiment, two random variables are defined as,

$$X = \begin{cases} 0, & \text{if the outcome is even,} \\ 1, & \text{if the outcome is odd,} \end{cases}$$

$$Y = \begin{cases} 0, & \text{if the outcome} \leq 3, \\ 1, & \text{if the outcome} > 3. \end{cases}$$

Also, a random variable, Z, is defined as $Z = 2X - Y$.

(a) Write down the joint probability mass function (PMF) of X and Y (in the form of a table, if you like).

(b) Write down the PMF of Z.

(c) Find $\mathbb{E}(Z)$, the expected value of Z.

MAT 271E – Probability and Statistics

Midterm Examination II

07.06.2014

Student Name : ______________________

Student Num. : ______________________

4 Questions, 100 Minutes

Please Show Your Work for Full Credit!

(25 pts) 1. Suppose X is a random variable whose probability density function (pdf) is given by

$$f_X(t) = \begin{cases} c\,t, & \text{if } t \in [0,4], \\ 0, & \text{if } t \notin [0,4]. \end{cases}$$

where 'c' is a constant.

(a) Determine c.

(b) Find and sketch $F_X(t)$, the cumulative distribution function (cdf) of X.

(c) What is the probability of the event $A = \{1 \le X \le 2\}$?

(25 pts) 2. Suppose X is a random variable uniformly distributed on $[-1, 3]$. Note that in this case, the pdf of X is given by

$$f_X(t) = \begin{cases} \frac{1}{4}, & \text{if } t \in [-1,3], \\ 0, & \text{if } t \notin [-1,3]. \end{cases}$$

Let $Z = |X|$.

(a) Find the probability of the event $\{Z \le 1\}$.

(b) Find $\mathbb{E}(Z)$, the expected value of Z.

(c) Find $f_Z(t)$, the pdf of Z.

(20 pts) 3. Suppose X and Y are independent random variables, whose pdfs are given as,

$$f_X(t) = \begin{cases} 2t, & \text{if } t \in [0,1], \\ 0, & \text{if } t \notin [0,1], \end{cases} \quad \text{and} \quad f_Y(t) = \begin{cases} 2-2t, & \text{if } t \in [0,1], \\ 0, & \text{if } t \notin [0,1]. \end{cases}$$

(a) Find $\mathbb{E}(X)$, the expected value of X.

(b) Find $\mathbb{E}(Y)$, the expected value of Y.

(c) Find the probability of the event $\{X \leq Y\}$.

(25 pts) 4. Suppose X, Y, Z are independent random variables. X and Y are continuous random variables with pdfs

$$f_X(t) = \begin{cases} 2t, & \text{if } t \in [0,1], \\ 0, & \text{if } t \notin [0,1], \end{cases} \quad \text{and} \quad f_Y(t) = \begin{cases} 1, & \text{if } t \in [0,1], \\ 0, & \text{if } t \notin [0,1]. \end{cases}$$

Z is a (discrete) Bernoulli random variable with PMF,

$$P_Z(k) = P\{Z = k\} = \begin{cases} \frac{1}{3}, & \text{if } k = 0, \\ \frac{2}{3}, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, U be a random variable defined as $U = Z\,X + (Z-1)\,Y$.

(a) Given that the event $A = \{Z = 1\}$ has occurred, find the expected value of U, that is $\mathbb{E}(U|A)$.

(b) Find $\mathbb{E}(U)$.

MAT 271E – Probability and Statistics

Final Examination

26.05.2014

Student Name : ______________________

Student Num. : ______________________

5 Questions, 120 Minutes

Please Show Your Work for Full Credit!

(15 pts) 1. Suppose there are two urns that contain blue and red balls.

- Urn-1 contains 2 red balls and 1 blue ball.
- Urn-2 contains 2 blue balls and 1 red ball.

Consider a random experiment as follows. A ball is randomly drawn from Urn-1, and placed in Urn-2. Then, a ball is randomly drawn from Urn-2.

(a) Given the event that the ball drawn from Urn-1 is blue, find the probability that the ball from Urn-2 is also blue.

(b) Compute the probability that the ball from Urn-2 is blue (given no information about the ball drawn from Urn-1).

(20 pts) 2. Suppose X and Y are two discrete random variables, whose joint probability mass function (PMF) is as given in the table below.

		X = 1	X = 2
Y	2	2/8	1/8
	3	3/8	2/8

Also, let $Z = X + Y$.

(a) Find the PMF of X.

(b) Find the joint PMF of Z and X. (You can make a table as above.)

(c) Let A be the event that $\{Z \geq 4\}$. Find the conditional expectation of X given A. That is, compute $\mathbb{E}(X|A)$.

(20 pts) 3. Let X be a continuous random variable. Suppose we are given that,

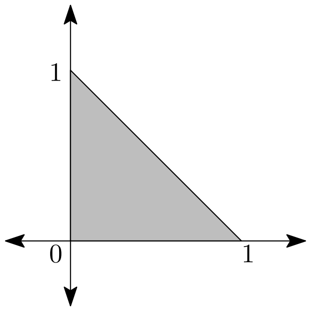
$$P\{t_1 \leq X \leq t_2\} = \begin{cases} \frac{t_2 - t_1}{(t_1+2)(t_2+2)}, & \text{if} \quad -1 \leq t_1 \leq t_2, \\ 0, & \text{if} \quad t_1 \leq t_2 < -1. \end{cases}$$

(a) Find the probability of the event $\{X \geq 0\}$.

(b) Find the probability of the event $\{|X| \leq 2\}$.

(c) Find $f_X(t)$, the probability density function (pdf) of X.

(20 pts) 4. Suppose X and Y are random variables, whose joint distribution is uniform inside the triangle between the points $(0,0)$, $(0,1)$, $(1,0)$ (as shown below).



(a) Find the probability of the event $\{t \leq Y\}$.

(b) Compute the probability of the event $\{t \leq X\} \cap \{t \leq Y\}$.

(c) Let $Z = \min(X, Y)$. Compute the probability of the event $\{Z \leq t\}$.

(25 pts) 5. Suppose X is a random variable, uniformly distributed $[0,1]$. That is, the probability density function (pdf) of X is given by,

$$f_X(t) = \begin{cases} 1, & \text{if} \quad t \in [0,1], \\ 0, & \text{if} \quad t \notin [0,1]. \end{cases}$$

Let $Y = \theta X$, where θ is a constant such that $\theta > 2$.

(a) Find $\mathbb{E}(Y)$, the expected value of Y (possibly in terms of θ).

(b) Find $\operatorname{var}(Y)$, the variance of Y (possibly in terms of θ).

(c) Find $f_Y(t)$, the pdf of Y (possibly in terms of θ).

(d) Compute the probability of the event $\{Y - 1 \leq \theta \leq Y + 1\}$ (possibly in terms of θ).