MAT 271E – Probability and Statistics

Spring 2015

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Class Meets : 13.30 – 16.30, Wednesday
EEB 5102

Office Hours : 10.00 – 12.00, Monday

Textbook : D. B. Bertsekas, J. N. Tsitsiklis, 'Introduction to Probability', 2^{nd} Edition, Athena-Scientific.

Grading : Homeworks (10%), 2 Midterms (25% each), Final (40%).

Webpage : There's a 'ninova' page. Please log in and check.

Tentative Course Outline

- Probability Space
  Probability models, conditioning, Bayes' rule, independence.
- Discrete Random Variables
  Probability mass function, functions of random variables, expectation, joint PMFs, conditioning, independence.
- General Random Variables
  Probability distribution function, cumulative distribution function, continuous Bayes' rule, correlation, conditional expectation.
- Limit Theorems
  Law of large numbers, central limit theorem.
- Introduction to Statistics
  Parameter estimation, linear regression, hypothesis testing.

MAT 271E – Homework 1

Due 18.02.2015

We randomly draw two cards from a deck. Recall that there are 4 colors in a deck (clubs, diamonds, hearts, spades) and 52 cards in total. Answer each question below independently. Think in terms of events. The idea is to make the computations simpler by introducing events.

1. Find the probability that both cards are hearts.
(Hint : Let $A_1 = \{\text{The first card is a heart}\}$, and $A_2 = \{\text{The second card is a heart}\}$. Observe that you are asked to compute $P(A_2 \cap A_1)$. Use conditioning!)

Solution. Let us define the events

$$A_1 = \{\text{The first card is a heart}\},$$
$$A_2 = \{\text{The second card is a heart}\}.$$

We need to find $P(A_2 \cap A_1)$. Note that

$$P(A_2 \cap A_1) = P(A_2|A_1)\, P(A_1).$$

But $P(A_2|A_1) = 12/51$ and $P(A_1) = 13/52$ (Why?). Thus we find $P(A_2 \cap A_1) = (13 \cdot 12)/(52 \cdot 51)$.

2. Find the probability that the second card is a heart.
(Hint : Consider A_1, A_2 as defined in the previous question's hint. You are asked to compute just $P(A_2)$ this time. Make use of A_1 to simplify computation.)

Solution. Note that

$$P(A_2) = P(A_2 \cap A_1) + P(A_2 \cap A_1^c).$$

We already know $P(A_2 \cap A_1)$ from Q1. Let us compute $P(A_2 \cap A_1^c)$. We have

$$P(A_2 \cap A_1^c) = P(A_2|A_1^c)\, P(A_1^c).$$

But $P(A_2|A_1^c) = 13/51$ and $P(A_1^c) = 39/52$ (Why?). Thus we find $P(A_2 \cap A_1) = (39 \cdot 13)/(52 \cdot 51)$. Thus we find,

$$P(A_2) = \frac{13 \cdot 12 + 39 \cdot 13}{52 \cdot 51} = \frac{1}{4}.$$

Note that $P(A_2) = P(A_1)$.

Here's an alternative solution, based on a symmetry argument discussed also in class. Define the events

$$C = \{\text{The second card is a club}\},$$
$$D = \{\text{The second card is a diamond}\},$$
$$H = \{\text{The second card is a heart}\},$$
$$S = \{\text{The second card is a spade}\}.$$

Observe that $C \cup D \cup H \cup S = \Omega$. Also, by symmetry, we expect the probability of each event to be equal. Therefore $P(C) = P(D) = P(H) = P(S) = 1/4$.

3. Find the probability that both cards have the same color.
(Hint : Define events to make your life easier. How is this question different from Q1?)

Solution. Let us define

$$B = \{\text{Both cards have the same color}\}. \tag{1}$$

Note that we are asked to compute $P(B)$. Now define the events

$$C = \{\text{Both cards are clubs}\},$$
$$D = \{\text{Both cards are diamonds}\},$$
$$H = \{\text{Both cards are hearts}\},$$
$$S = \{\text{Both cards are spades}\}.$$

Note that $B = C \cup D \cup H \cup S$. Since these events are disjoint (i.e. $C \cap D = \emptyset$, $C \cap H = \emptyset$, etc.), we have $P(B) = P(C) + P(D) + P(H) + P(S)$. But by symmetry, we have $P(C) = P(D) = P(H) = P(S)$. Also, we already computed in Q1 that $P(H) = 12/(4 \cdot 51)$. Therefore, $P(B) = 12/51$.

4. Find the probability that at least one of the cards is a heart.
(Hint : Note that at least one means one or two in this case. Define simple events relevant for the problem and express the event of interest in terms of the events you defined.)

Solution. Let us define the events

$$\begin{aligned} A &= \{\text{At least one of the cards is a heart}\}, \\ A_1 &= \{\text{The first card is a heart}\}, \\ A_2 &= \{\text{The second card is a heart}\}. \end{aligned}$$

Note that we are asked to compute $P(A)$ and $A = A_1 \cup A_2$. But $A_1 \cap A_2 \neq \emptyset$, so we cannot add the probabilities of A_1 and A_2 to obtain the probability of A. To write A in terms of disjoint sets, note that $A = A_1 \cup \big(A_2 \cap A_1^c\big)$. We can compute easily that $P(A_1) = 13/52$. Notice also that

$$A_1^c = \{\text{The first card is not a heart}\},$$

so $P(A_1^c) = 39/52$. Now,

$$P(A_2 \cap A_1^c) = P(A_2|A_1^c)\, P(A_1^c) = \frac{13}{51}\frac{39}{52}.$$

We finally obtain,

$$P(A) = P(A_1) + P(A_2 \cap A_1^c) = \frac{13}{52} + \frac{13}{51}\frac{39}{52} = \frac{20}{51}.$$

5. Given that both cards are aces, find the probability that one of them is a heart.

Solution. Define the events

$$\begin{aligned} A &= \{\text{Both cards are aces}\}, \\ B &= \{\text{One of the cards is a heart}\}. \end{aligned}$$

We need to compute $P(B|A)$. But note that the set $A \cap B$ is so small, we can list its elements. Suppose (h_a, s_a) denotes the outcome for which the first card is the ace of hearts and the second card is the ace of spades, etc. Then,

$$A \cap B = \{(h_a, c_a), (h_a, d_a), (h_a, s_a), (c_a, h_a), (d_a, h_a), (s_a, h_a)\}.$$

Since the probability of each element in $A \cap B$ is $1/(52 \cdot 51)$, we find $P(A \cap B) = 6/(52 \cdot 51)$. Also, since $P(A) = (4 \cdot 3)/(52 \cdot 51)$ (why?), we find

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{6}{52 \cdot 51}\,\frac{52 \cdot 51}{4 \cdot 3} = \frac{1}{2}.$$

6. Given that both cards are faces (i.e., jack, queen or king), find the probability that both are hearts.

Solution. Let us define the events

$$\begin{aligned} F_1 &= \{\text{The first card is a face}\}, \\ F_2 &= \{\text{The second card is a face}\}, \\ H_1 &= \{\text{The first card is a heart}\}, \\ H_2 &= \{\text{The second card is a heart}\}. \end{aligned}$$

Note that we are asked to compute $P(H_1 \cap H_2|F_1 \cap F_2)$. Note that

$$P(H_1 \cap H_2|F_1 \cap F_2) = \frac{P\Big(\big(H_1 \cap F_1\big) \cap \big(H_2 \cap F_2\big)\Big)}{P(F_1 \cap F_2)}.$$

Notice that (why?)

$$P(F_1 \cap F_2) = P(F_2|F_1)\, P(F_1) =,$$

and (why?)

$$P\Big(\big(H_1 \cap F_1\big) \cap \big(H_2 \cap F_2\big)\Big) = P\Big(\big(H_2 \cap F_2\big)\Big|\big(H_1 \cap F_1\big)\Big)\, P(H_1 \cap F_1) = \frac{2}{51}\frac{3}{52}.$$

Thus we find

$$P(H_1 \cap H_2|F_1 \cap F_2) = \frac{2}{51}\frac{3}{52}\frac{51}{11}\frac{52}{12} = \frac{1}{22}.$$

MAT 271E – Homework 2

Due 04.03.2015

1. We randomly select a card from a deck. Consider the following events.

$$\begin{aligned} A &= \{\text{the card is a heart}\}, \\ B &= \{\text{the card is an ace}\}, \\ C &= \{\text{the card is a king}\}. \end{aligned}$$

Check whether the following pairs of events are independent or not.

(a) A and B.
(b) A and C.
(c) B and C.
(d) B and B^c.
(e) B^c and C^c.

Solution. Note that $P(A) = 1/4$, $P(B) = 1/13$, $P(C) = 1/13$.

(a) We observe that

$$P(A \cap B) = 1/52 = P(A)\,P(B).$$

Therefore A and B are independent.

(b) Same reasoning as in (a) shows that A and C are independent.

(c) Note that $B \cap C = \emptyset$ but $P(B) > 0$, $P(C) > 0$. Thus B and C are not independent.

(d) Same reasoning as in (d). An event and its complement are not independent unless one of them has zero probability.

(e) Note that we have $B^c \cap C^c \neq \emptyset$ so we cannot reason as in (c). We could check the independence condition as in (a), but since we know that B and C are independent, here's a more elegant argument that makes use of this and another useful fact (for this question, it is of course trivial to check the independence condition but in other scenarios, an argument like the one below might be handy).

Here's the useful fact : if two arbitrary events E and F are independent, then E and F^c are also independent. To see this, observe that if E and F are independent,

$$\begin{aligned} P(E \cap F^c) &= P(E) - P(E \cap F) \\ &= P(E) - P(E)\,P(F) \\ &= P(E)\,(1 - P(F)) \\ &= P(E)\,P(F^c) \end{aligned}$$

from which the claim follows.

Suppose now that B^c and C^c are independent. Then B^c and C are also independent. But then B and C must also be independent, and we know that this is not true. Therefore the initial assumption about the independence of B^c and C^c must be wrong.

(Observe that actually an equivalent statement of the useful fact above is that if E and F are not independent, then E and F^c are not independent either.)

2. Suppose that three events satisfy the relation $A \subset B \subset C$. Show that $P(A|B) \geq P(A|C)$. (What does this mean intuitively?)

Solution. Note that $A \cap B = A \cap C$ and $P(C) \geq P(B)$. Thus,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \geq \frac{P(A \cap C)}{P(C)} = P(A|C).$$

3. There are thirty balls in an urn and three of them are red. The balls are randomly distributed to three people so that each person gets ten balls. What is the probability that each person has a red ball?

Solution. While you can solve this using combinatorics, there's an alternative solution that makes use of events to simplify the argument.

(Soln.1) Let us call label the persons as P_1, P_2, P_3. Let us define the events,

$$\begin{aligned}
A &= \{P_1 \text{ gets a single red ball}\},\\
B &= \{P_2 \text{ gets a single red ball}\},\\
C &= \{P_3 \text{ gets a single red ball}\},\\
D &= \{\text{each person gets a single red ball}\}.
\end{aligned}$$

Note that we are asked to compute $P(D)$. Notice also that $D = A \cap B \cap C$. However we also have, $A \cap B \subset C$. Therefore, $D = A \cap B$. To compute $P(A \cap B)$, we will use the identity $P(A \cap B) = P(B|A)\,P(A)$. Let us start with $P(A)$. Let us further define the events

$$\begin{aligned}
A_1 &= \{\text{the first ball of } P_1 \text{ is red, the rest are not}\},\\
A_2 &= \{\text{the second ball of } P_1 \text{ is red, the rest are not}\},\\
&\;\;\vdots\\
A_{10} &= \{\text{the tenth ball of } P_1 \text{ is red, the rest are not}\}.
\end{aligned}$$

Note that $A = A_1 \cup A_2 \cup \cdots \cup A_{10}$, and $A_k \cap A_j = \emptyset$ if $k \neq j$. Therefore, $P(A) = \sum_k P(A_k)$. But we have (why?)

$$P(A_k) = \frac{3}{30}\,\frac{27}{29}\,\frac{26}{28}\cdots\frac{19}{21}$$

for any $1 \leq k \leq 10$. Thus,

$$P(A) = \frac{27 \cdot 26 \cdots 19}{29 \cdot 28 \cdots 21}.$$

Let us now compute $P(B|A)$. Let us similarly define the events

$$\begin{aligned}
B_1 &= \{\text{the first ball of } P_2 \text{ is red, the rest are not}\},\\
B_2 &= \{\text{the second ball of } P_2 \text{ is red, the rest are not}\},\\
&\;\;\vdots\\
B_{10} &= \{\text{the tenth ball of } P_2 \text{ is red, the rest are not}\}.
\end{aligned}$$

We have $B = B_1 \cup B_2 \cup \cdots \cup B_{10}$, and $B_k \cap B_j = \emptyset$, if $k \neq j$. Therefore, it follows that

$$\begin{aligned}
P(B|A) &= \frac{P\big((B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_{10} \cap A)\big)}{P(A)}\\
&= \frac{P\big(B_1 \cap A\big)}{P(A)} + \frac{P\big(B_2 \cap A\big)}{P(A)} + \cdots + \frac{P\big(B_{10} \cap A\big)}{P(A)}\\
&= \sum_{k=1}^{10} P(B_k|A).
\end{aligned}$$

Note that given A, there are 20 balls available to P_2, two of which are red. Therefore, we have,

$$P(B_k|A) = \frac{2}{20}\,\frac{18}{19}\,\frac{17}{18}\cdots\frac{10}{11}.$$

for any $1 \leq k \leq 10$. Thus,

$$P(B|A) = \frac{18 \cdot 17 \cdots 10}{19 \cdot 18 \cdots 11}.$$

Combining we obtain,

$$P(A \cap B) = P(B|A)\,P(A) = \frac{27 \cdot 26 \cdots 10}{(29 \cdot 28 \cdots 21) \cdot (19 \cdot 18 \cdots 11)} = \frac{20 \cdot 10}{29 \cdot 28}.$$

(Soln.2) Suppose we draw the balls one by one and form a sequence of length 30. The first ten balls are given to P_1 (first person), the second ten balls are given to P_2 and the last ten balls are given to P_3 (if you think the order with which the balls are handed in matters, you could first form a sequence of length 30 as above, and then give the first ten to one of the persons chosen randomly, the second ten to another chosen randomly etc. – but it does not matter in fact). Suppose we also label the red balls as R_1, R_2, R_3. These red balls can be placed in a total of $\binom{30}{3} \cdot 3!$ different arrangements. But we are interested in the number of distributions for which there is a red ball placed between positions 1 to 10, another between positions 11 to 20 and another between 21 to 30. The total number of arrangements under such a restriction is $\big(10 \cdot 10 \cdot 10 \cdot 3!\big)$. The ratio gives us the probability we are after,

$$\frac{10 \cdot 10 \cdot 10 \cdot 3!}{\binom{30}{3} \cdot 3!} = \frac{20 \cdot 10}{29 \cdot 28}.$$

4. Consider an experiment that consists of three independent tosses of a coin. Note that we can list the sample space as $\Omega = \{HHH, HHT, HTH, HTT, \dots, TTT\}$. Let X be a discrete random variable that maps Ω to real numbers, defined as $X(\omega) =$ (number of heads in ω) $-$ (number of tails in ω), where $\omega \in \Omega$. Also let $Y = 3X - 3$.

(a) Write down the largest set A such that $X(\omega) \le 0$ if $\omega \in A$.

(b) Write down the largest set B such that $Y(\omega) \le -1$ if $\omega \in B$.

Solution. (a) $A = \{HTT, THT, TTH, TTT\}$.

(b) $B = \{\omega : 3X(\omega) - 3 \le -1\} = \{\omega : X(\omega) \le 2/3\} = A$.

MAT 271E – Homework 3

Due 18.03.2015

1. Suppose X is a geometric random variable with PMF given as

$$P_X(k) = \begin{cases} (1-p)^{k-1}\,p, & \text{if } k \text{ is a positive integer,} \\ 0, & \text{otherwise.} \end{cases}$$

Also, let Y and Z be random variables defined as,

$$Y = \begin{cases} 1, & \text{if } X \text{ is even,} \\ -1, & \text{if } X \text{ is odd,} \end{cases}$$

$$Z = \begin{cases} 0, & \text{if } X \leq 10, \\ 1, & \text{if } 10 < X. \end{cases}$$

(a) Find the PMF of Y.

(b) Compute $\mathbb{E}(Z)$.

Solution. (a) We compute

$$\begin{aligned} P(\{Y=1\}) &= \sum_{k=1}^{\infty}(1-p)^{2k-1}\,p \\ &= \frac{p}{1-p}\sum_{k=1}^{\infty}\Big((1-p)^2\Big)^k \\ &= \frac{p}{1-p}\,\frac{(1-p)^2}{2p-p^2} \\ &= \frac{1-p}{2-p} \end{aligned}$$

Since Y can only take two values, we have, $P(\{Y=-1\}) = 1 - P(\{Y=1\})$. Thus,

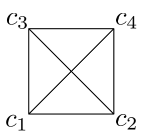
$$P_Y(k) = \begin{cases} \frac{1}{2-p}, & \text{if } k=-1, \\ \frac{1-p}{2-p}, & \text{if } k=1, \\ 0 & \text{othwerwise.} \end{cases}$$

(b) It is easier to compute $\mathbb{E}(Z)$ using P_X here.

$$\begin{aligned} \mathbb{E}(Z) &= \sum_{k=11}^{\infty} P_X(k) \\ &= \sum_{k=11}^{\infty}(1-p)^{k-1}\,p \\ &= \frac{(1-p)^{10}}{p}\sum_{k=0}^{\infty}(1-p)^k \\ &= \frac{(1-p)^{10}}{p^2}. \end{aligned}$$

Note that $\mathbb{E}(Z) = P(\{X > 10\})$.

2. Consider a square whose corners are labeled as c_i (see below). A particle is placed on one of the corners and starts moving from one corner to another connected by an edge at each step (note that there is an edge between c_1 and c_4 and an edge between c_2 and c_3).



If the particle reaches c_4, it is trapped and stops moving. Assume that the steps taken by the particle are independent and the particle chooses its next stop randomly (i.e., all choices are equally likely). Suppose also that the particle is initially placed at c_1. Also, let X denote the total number of steps taken by the particle (to reach c_4).

(a) Find the probability that $X = 1$.
(b) Find the probability that $X = 2$.
(c) Find the PMF of X.
(d) Compute the expected value of X.

Solution. (a) $P(\{X = 1\} = 1/3$ since the probability of going to c_4 from c_1 is $1/3$.

(b) In this case, we need to compute the probability of a sequence of the form $d\,c_4$ where d is either c_2 or c_3. Thus $P(\{X = 2\}) = \frac{2}{3}\frac{1}{3}$.

(c) Note that by similar reasoning $X = n$ occurs if we observe a sequence of the form $d_1\,d_2\,\ldots d_{n-1}\,c_4$, where d_k's are different than c_4. Thus the PMF is given by

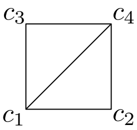
$$P_X(n) = \frac{2^{n-1}}{3^n},$$

where n is a positive integer.

(d) Using $P_X(n)$, we compute,

$$\mathbb{E}(X) = \frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^{n-1} = 3.$$

3. Repeat the question above but this time assume that the corners are connected as shown below.



Notice that now c_2 and c_3 are not connected. Consequently, if the particle is at c_2, the probability of choosing c_1 or c_4 as the next stop is $1/2$.

Solution. (a) $P(\{X = 1\} = 1/3$ since the probability of going to c_4 from c_1 is $1/3$.

(b) $X = 2$ if and only if one of the corner sequences in $\{(c_2\,c_4), (c_3\,c_4)\}$ occurs. Therefore,

$$P(\{X = 2\} = \frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{1}{2} = \frac{1}{3}.$$

(c) Note that $X = 4$ occurs if and only if we observe a corner sequence of the form $(x\,c_1\,c_4)$, where x is either c_2 or c_3. That is, we have to come back to c_1 if $X > 2$. More generally, $X = 2k + 1$ occurs if and only if a sequence of the form

$$(x\,c_1\,x\,c_1\,\ldots\,x\,c_1\,c_4)$$

occurs, where x is either c_2 or c_3. Note that the number of x's is equal to k in that sequence. The number of c_1's is also k. Thus we compute

$$P(\{X = 2k + 1\} = \left(\frac{2}{3}\right)^k \left(\frac{1}{2}\right)^k \frac{1}{3} = \left(\frac{1}{3}\right)^{k+1}$$

By similar reasoning, we can observe that $X = 2k$ (with $k > 1$) if and only if a sequence of the form

$$(x\,c_1\,x\,c_1\,\ldots\,x\,c_4)$$

occurs. Here, the number of x's is k and the number of c_1's is $k - 1$. Thus,

$$P(\{X = 2k\} = \left(\frac{2}{3}\right)^k \left(\frac{1}{2}\right)^{k-1} \frac{1}{2} = \left(\frac{1}{3}\right)^k$$

Therefore, the PMF is given by

$$P_X(k) = \begin{cases} \left(\frac{1}{3}\right)^{(k+1)/2}, & \text{if } k \text{ is an odd integer,} \\ \left(\frac{1}{3}\right)^{k/2}, & \text{if } k \text{ is an even integer.} \end{cases}$$

(d) Using the PMF, we can now compute

$$\begin{aligned} \mathbb{E}(X) &= \sum_{k=1}^{\infty} k\,P_X(k) \\ &= \sum_{n=0}^{\infty} (2n+1)\left(\frac{1}{3}\right)^{n+1} + \sum_{n=1}^{\infty} (2n)\left(\frac{1}{3}\right)^n = 5/2 \end{aligned}$$

MAT 271E – Homework 4

Due 25.03.2015

1. Suppose X, Y are independent random variables (integer valued) with PMF given as

$$P_X(k) = \begin{cases} 1/2, & \text{if } k = 0, \\ 1/2, & \text{if } k = 1, \end{cases} \quad \text{and } P_Y(k) = \left(\frac{3}{4}\right)^{k-1} \frac{1}{4}, \text{ if } k \geq 1.$$

(a) Let $Z = X + Y$. Find the PMF of Z.

(b) Let $Z = Y - X$. Find the PMF of Z.

Solution. (a) Note that

$$\{Z = 1\} = \{X = 0, Y = 1\},$$
$$\{Z = k\} = \{X = 0, Y = k\} \cup \{X = 1, Y = k-1\}, \text{ for } k > 1.$$

Thanks to the independence of X and Y, we thus have

$$P_Z(k) = \begin{cases} \frac{1}{2} \cdot \frac{1}{4}, & \text{if } k = 1, \\ \frac{1}{2} \cdot \left(\frac{3}{4}\right)^{k-1} \frac{1}{4} + \frac{1}{2} \cdot \left(\frac{3}{4}\right)^{k-2} \frac{1}{4}, & \text{if } k > 1. \end{cases}$$

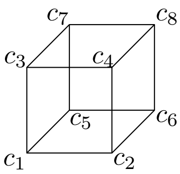
(b) We similarly observe that

$$\{Z = 0\} = \{X = 1, Y = 1\},$$
$$\{Z = k\} = \{X = 0, Y = k\} \cup \{X = 1, Y = k+1\}, \text{ for } k > 0.$$

Thanks to the independence of X and Y, we thus have

$$P_Z(k) = \begin{cases} \frac{1}{2} \cdot \frac{1}{4}, & \text{if } k = 0, \\ \frac{1}{2} \cdot \left(\frac{3}{4}\right)^{k-1} \frac{1}{4} + \frac{1}{2} \cdot \left(\frac{3}{4}\right)^{k} \frac{1}{4}, & \text{if } k > 0. \end{cases}$$

2. Consider a box whose corners are labeled as c_i (see below).



A particle is placed on c_1 and starts moving from one corner to another connected by an edge at each step. If the particle reaches the opposite corner c_8, it is trapped and stops moving. Assume that the steps taken by the particle are independent and the particle chooses its next stop randomly (i.e., all three choices are equally likely). Also, let X denote the total number of steps taken by the particle (to reach c_8). Our goal is to compute the expected value of X. The steps below propose an approach based on conditional expectations.

For conditioning, let us also define the random variable Y_n as the minimum number of steps that needs to be taken to reach c_8 after the n^{th} step. Observe that $Y_0 = 3$ and $Y_1 = 2$ with probability one.

Let us also define $d_1 = \mathbb{E}(X|Y_n = 1) - n$, $d_2 = \mathbb{E}(X|Y_n = 2) - n$, $d_3 = \mathbb{E}(X|Y_n = 3) - n$. Notice that d_k is independent of n. d_k is the expected number of steps after we observe that at the n^{th} step, we are at least k steps away from c_8. Observe also that $d_3 = 1 + d_2$ and $\mathbb{E}(X) = d_3$.

(a) Find the conditional probabilities $P(Y_n = 3|Y_{n-1} = 2)$, $P(Y_n = 2|Y_{n-1} = 1)$, $P(Y_n = 1|Y_{n-1} = 2)$.

(b) Show that

$$P\Big((X = 2n+3)\Big|(X \geq 2n+1)\Big) = \frac{7}{9}\, P\Big((X = 2n+1)\Big|(X \geq 2n+1)\Big).$$

(c) Show using part (b) that $\mathbb{E}(X) < \infty$.

(d) We noted that $d_3 = 1 + d_2$. Find a different equation that expresses d_2 in terms of d_1 and d_3. Then find yet another independent equation that expresses d_1 in terms of d_2. Solve the three equations to obtain d_1. (Why do we need part (c)?)

Solution. (a) $P(Y_n = 3|Y_{n-1} = 2) = 1/3$, $P(Y_n = 1|Y_{n-1} = 2) = 2/3$ because every corner two steps away from c_8 is adjacent to c_1 and two other corners which are one-step away from c_8.

$P(Y_n = 2|Y_{n-1} = 1) = 2/3$ because every corner which is one-step away from c_8 is a neighbor of two other corners which are two steps away from c_8.

$P(Y_n = 2|Y_{n-1} = 3) = 1$ because all neighbors of c_1 are two steps away from c_8.

(b) Observe that Xcannot be negative and if $X = 2n + 1$, then the realization of the sequence obtained by Y_n's is of the form

$$(y_1, y_2, y_3, y_4, \ldots, y_{2n-1}, 1, 0),$$

where $y_{2k+1} = 2$, for $0 \le k < n$. Now if $A = \{X \ge 2n + 1\}$ occurs, this means that $y_k \ne 0$ for $k \le 2n$ and $y_{2n-1} = 2$.

Given A, the event $B = \{X = 2n+1\}$ occurs if and only if $(y_{2n}, y_{2n+1}) = (1, 0)$. Therefore $P(B|A) = (2/3) \cdot (1/3)$.

Given A, the event $C = \{X = 2n + 3\}$ occurs if and only if

$$(y_{2n}, y_{2n+1}, y_{2n+2}, y_{2n+3}) \in \{(1, 2, 1, 0), (3, 2, 1, 0)\}.$$

Therefore,

$$P(C|A) = (2/3) \cdot (1/3) \cdot (2/3) \cdot (1/3) + (1/3) \cdot 1 \cdot (2/3) \cdot (1/3).$$

Computing ratios, we find that $P(C|A) = (2/9)\, P(B|A)$.

Observe now that actually, $B \subset A$, $C \subset A$. Therefore, $P(C) = (2/9)P(B)$ (show this!).

(Note : You can actually derive the PMF and obtain $\mathbb{E}(X)$ directly.)

(c) It follows from part (b) that $P_X(2k) = 0$ and $P_X(2k + 1) \le (7/9)^k$. Thus,

$$\mathbb{E}(X) \le \sum_{k=0}^{\infty}(2k + 1)\,(7/9)^k < \infty.$$

(d) Let $A = \{Y_n = 2\}$, $B_1 = \{Y_{n+1} = 1\}$, $B_3 = \{Y_{n+1} = 3\}$. Then, we can write, $A = (A \cap B_1) \cup (A \cap B_3)$, where $(A \cap B_1)$ and $(A \cap B_3)$ are disjoint. We now note that (show this!)

$$\mathbb{E}(X|A) = \mathbb{E}(X|A \cap B_1)\, P(B_1|A) + \mathbb{E}(X|A \cap B_3)\, P(B_3|A).$$

But notice that if we know Y_{n+1}, then the number of remaining steps no longer depends on the value of Y_n. Therefore,

$$\mathbb{E}(X|A \cap B_1) = \mathbb{E}(X|B_1) = d_1 + n + 1.$$

Similarly,

$$\mathbb{E}(X|A \cap B_3) = \mathbb{E}(X|B_3) = d_3 + n + 1.$$

Noting that $P(B_1|A) = 2/3$, $P(B_3|A) = 1/3$ (by part (a)), we thus obtain,

$$\mathbb{E}(X|A) = d_2 + n = \frac{2}{3}\,(d_1 + n + 1) + \frac{1}{3}\,(d_3 + n + 1).$$

Rearranging, we obtain,

$$d_2 = \frac{2}{3}\,d_1 + \frac{1}{3}\,d_3 + 1.$$

By a similar reasoning, we obtain another equality as

$$d_1 = \frac{2}{3}\,d_2 + 1.$$

Putting the three equations together, we obtain the system of equations,

$$\begin{aligned} d_1 &= \frac{2}{3}\,d_2 + 1, \\ d_2 &= \frac{2}{3}\,d_1 + \frac{1}{3}\,d_3 + 1, \\ d_3 &= d_2 + 1. \end{aligned}$$

This is a linear system. Solving for d_i's, we find $d_1 = 7$, $d_2 = 9$, $d_3 = 10$. This is the only finite solution, observe also that $d_i = \infty$also satisfies the equations – but we ruled out this case in part (c). Thus $\mathbb{E}(X) = 10$.

MAT 271E – Homework 5

Due 08.04.2015

In the questions below, suppose X_1, X_2 and X_3 are independent random variables uniformly distributed on $[0, 1]$.

1. Compute the probability that $X_1 \leq X_2$.

   Solution. Let us define the events

$$A_1 = \{X_1 \leq X_2\}$$
$$A_2 = \{X_2 \leq X_1\}.$$

   Note that the intersection of A_i's has zero probability and $A_1 \cup A_2 = \Omega$. Also, by symmetry, $P(A_1) = P(A_2)$. Therefore, $P(A_1) = P(A_2) = 1/2$.

   An alternative solution is as follows :

$$\begin{aligned} P(A_1) &= \int_{-\infty}^{\infty} P(A_1 | X_2 = t)\, f_{X_2}(t)\, dt \\ &= \int_0^1 P(X_1 \leq t)\, dt \\ &= \int_0^1 t\, dt = 1/2. \end{aligned}$$

2. Compute the probability that $X_1 \leq 2X_2$.

   Solution.

$$\begin{aligned} P(X_1 \leq 2X_2) &= \int_{-\infty}^{\infty} P(X_1 \leq 2X_2 | X_2 = t)\, f_{X_2}(t)\, dt \\ &= \int_0^1 P(X_1 \leq 2t)\, dt. \end{aligned}$$

   Observe now that

$$P(X_1 \leq 2t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 1/2, \\ 1, & \text{if } 1/2 < t \leq 1. \end{cases}$$

   Thus,

$$P(X_1 \leq 2X_2) = \int_0^{1/2} 2t\, dt + \int_{1/2}^1 1\; dt = \frac{3}{4}.$$

   An alternative solution can be obtained by conditioning on $X_1 = t$ and $X_2 = u$.

$$\begin{aligned} P(X_1 \leq 2X_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(X_1 \leq 2X_2 | X_1 = t, X_2 = u)\, f_{X_1,X_2}(t,u)\, du\, dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(t \leq 2u)\, f_{X_1,X_2}(t,u)\, du\, dt \\ &= \int_0^1 \int_0^1 P(t \leq 2u)\, du\, dt. \end{aligned}$$

   Observe now that for fixed numbers t and u,

$$P(t \leq 2u) = \begin{cases} 1, & \text{if } t/2 \leq u, \\ 0, & \text{if } t > 2u. \end{cases}$$

   Therefore,

$$\begin{aligned} P(X_1 \leq 2X_2) &= \int_0^1 \int_{t/2}^1 1\, du\, dt \\ &= \int_0^1 (1 - t/2)\, dt \\ &= \frac{3}{4}. \end{aligned}$$

3. Compute the probability that $X_1 \le X_2 \le X_3$.

   Solution. With a symmetry argument as in Q1, we can see that this probability is 1/6. But let us see how to compute this probability by conditioning. We have,

$$\begin{aligned} P(X_1 \le X_2 \le X_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(X_1 \le X_2 \le X_3 | X_2 = t, X_3 = u)\, f_{X_2,X_3}(t,u)\, du\, dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(X_1 \le t \le u)\, f_{X_1,X_2}(t,u)\, du\, dt \\ &= \int_0^1\int_0^1 P(X_1 \le t \le u)\, du\, dt. \end{aligned}$$

   Observe that for fixed t and u with $0 \le t \le 1$, $0 \le u \le 1$,

$$P(X_1 \le t \le u) = \begin{cases} t, & \text{if } t \le u, \\ 0, & \text{if } u > t. \end{cases}$$

   Therefore,

$$\begin{aligned} P(X_1 \le X_2 \le X_3) &= \int_0^1\int_t^1 t\, du\, dt \\ &= \int_0^1 t(1-t)\, dt \\ &= \frac{1}{6}. \end{aligned}$$

4. Compute the probability that $X_1 \le 2X_2 \le 3X_3$.

   Solution. We have,

$$\begin{aligned} P(X_1 \le 2X_2 \le 3X_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(X_1 \le 2X_2 \le 3X_3 | X_2 = t, X_3 = u)\, f_{X_2,X_3}(t,u)\, du\, dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(X_1 \le 2t \le 3u)\, f_{X_2,X_3}(t,u)\, du\, dt \\ &= \int_0^1\int_0^1 P(X_1 \le 2t \le 3u)\, du\, dt. \end{aligned}$$

   Observe that for fixed t and u with $0 \le t \le 1$, $0 \le u \le 1$,

$$P(X_1 \le 2t \le 3u) = \begin{cases} 2t, & \text{if } 0 \le t \le 1/2 \text{ and } 2t/3 \le u, \\ 1, & \text{if } 1/2 \le t \text{ and } 2t/3 \le u, \\ 0, & \text{if } u > 2t/3. \end{cases}$$

   Therefore,

$$\begin{aligned} P(X_1 \le X_2 \le X_3) &= \int_0^{1/2}\int_{2t/3}^1 2t\, du\, dt + \int_{1/2}^1\int_{2t/3}^1 1\, du\, dt \\ &= \int_0^{1/2} 2t\,(1 - 2t/3)\, dt + \int_{1/2}^1 (1 - 2t/3)\, dt \\ &= (t^2 - 4t^3/9)\Big|_0^{1/2} + (t - t^2/3)\Big|_{1/2}^1 \\ &= \frac{4}{9}. \end{aligned}$$

5. Using X_i's suppose we form a symmetric matrix as,

$$A = \begin{bmatrix} X_1 & X_3 \\ X_3 & X_2 \end{bmatrix}.$$

   Note that A is a 2×2 symmetric matrix with positive entries.

(a) Compute the probability that A is invertible.

(b) In this particular setting, A is positive definite if its determinant is positive. Compute the probability that A is positive definite.
Hint : For $t \in [0,1]$, what is the probability that $X_1 X_2 \geq t$?
(Note : You should know the definition of positive definiteness from linear algebra – if you don't remember please look it up!)

Solution. (a) A is invertible if and only if its determinant is non-zero or, if the event $\{X_1 X_2 \neq X_3^2\}$ occurs. Note that

$$\begin{aligned} P(X_1 X_2 \neq X_3^2) &= \int_{-\infty}^{\infty} P(X_1 X_2 \neq X_3^2 | X_3 = t)\, f_{X_3}(t)\, dt \\ &= \int_{-\infty}^{\infty} P(X_1 X_2 \neq t^2)\, f_{X_3}(t)\, dt. \end{aligned}$$

But $P(X_1 X_2 \neq t^2) = 1$ for any t. Therefore, $P(X_1 X_2 \neq X_3^2) = 1$. Thus A is invertible with probability one.

(b) We have,

$$\begin{aligned} P(X_1 X_2 \geq X_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(X_1 X_2 \geq X_3 | X_2 = t, X_3 = u)\, f_{X_2,X_3}(t,u)\, du\, dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(X_1 \geq t/u)\, f_{X_2,X_3}(t,u)\, du\, dt \\ &= \int_0^1\int_0^1 P(X_1 \geq t/u)\, du\, dt. \end{aligned}$$

Observe that for fixed t and u with $0 \leq t \leq 1$, $0 \leq u \leq 1$,

$$P(X_1 \geq t/u) = \begin{cases} (1 - t/u), & \text{if } t \leq u, \\ 0, & \text{if } t > u. \end{cases}$$

Therefore (check this!),

$$\begin{aligned} P(X_1 X_2 \geq X_3) &= \int_0^1\int_t^1 1 - \frac{t}{u}\, du\, dt \\ &= \int_0^1 (1-t) + t\,\ln(t)\, dt \\ &= \frac{1}{4}. \end{aligned}$$

6. Let $Z = 2X_1$. Write down the cdf and the pdf of Z.

Solution. We compute

$$P(Z \leq t) = P(X_1 \leq 2t) = F_X(2t) = \begin{cases} 0, & \text{if } t < 0, \\ 2t, & \text{if } 0 \leq t \leq 1/2, \\ 1, & \text{if } 1/2 < t. \end{cases}$$

Differentiating with respect to t, we obtain,

$$f_Z(t) = F_Z'(t) = \begin{cases} 0, & \text{if } t < 0, \\ 2, & \text{if } 0 \leq t \leq 1/2, \\ 0, & \text{if } 1/2 < t. \end{cases}$$

7. Let $Z = -X_1$. Write down the cdf and the pdf of Z.

Solution. We compute

$$\begin{aligned} P(Z \leq t) &= P(-X_1 \leq t) \\ &= P(-t \leq X_1) \\ &= 1 - P(X_1 < -t) \\ &= 1 - F_X(-t) \\ &= \begin{cases} 0, & \text{if } t < -1, \\ 1 + t, & \text{if } -1 \leq t \leq 0, \\ 1, & \text{if } 0 < t. \end{cases} \end{aligned}$$

Differentiating with respect to t, we obtain,

$$f_Z(t) = F_Z'(t) = \begin{cases} 0, & \text{if } t < -1, \\ 1, & \text{if } -1 \le t \le 0, \\ 0, & \text{if } 0 < t. \end{cases}$$

8. Let $Z = X_1 + X_2$. Write down the cdf and the pdf of Z.

Solution. We compute

$$\begin{aligned} P(Z \le t) &= P(X_1 + X_2 \le t) \\ &= \int_{-\infty}^{\infty} P(X_1 + X_2 \le t | X_2 = u)\, f_{X_2}(u)\, du \\ &= \int_{-\infty}^{\infty} P(X_1 \le t - u)\, f_{X_2}(u)\, du \\ &= \int_0^1 P(X_1 \le t - u)\, du. \end{aligned}$$

Notice now that

$$P(X_1 \le t - u) = \begin{cases} 0, & \text{if } t - u < 0, \\ t - u, & \text{if } 0 \le t - u \le 1, \\ 1, & \text{if } 1 < t - u. \end{cases}$$

Thus the form of the integral depends on the value of t. Let us investigate that

- If $t < 0$, for $u \in [0, 1]$, we have $t - u < 0$. Thus,

  $$P(Z \le t) = \int_0^1 P(X_1 \le t - u)\, du = 0.$$

- If $0 \le t \le 1$, for $u \in [0, 1]$,

  $$P(X_1 \le t - u) = \begin{cases} 0, & \text{if } t < u, \\ t - u, & \text{if } u \le t. \end{cases}$$

  Thus

  $$P(Z \le t) = \int_0^t (t - u)\, du = t^2/2.$$

- If $1 \le t \le 2$, for $u \in [0, 1]$,

  $$P(X_1 \le t - u) = \begin{cases} t - u, & \text{if } t - 1 < u, \\ 1, & \text{if } t - 1 > u. \end{cases}$$

  Thus

  $$P(Z \le t) = \int_0^{t-1} 1\, du + \int_{t-1}^1 t - u\, du = 2t - \frac{t^2}{2} - 1.$$

- If $t > 2$, for $u \in [0, 1]$, we have $t - u > 1$. Thus,

  $$P(Z \le t) = \int_0^1 P(X_1 \le t - u)\, du = \int_0^1 1\, du = 1.$$

To summarize, the cdf and the pdf of Z is given as,

$$F_Z(t) = P(Z \le t) = \begin{cases} 0, & \text{if } t < 0, \\ t^2/2, & \text{if } 0 \le t \le 1, \\ 2t - t^2/2 - 1, & \text{if } 1 \le t \le 2, \\ 1, & \text{if } 2 < t, \end{cases}$$

$$f_Z(t) = F_Z'(t) = \begin{cases} 0, & \text{if } t < 0, \\ t, & \text{if } 0 \le t \le 1, \\ 2 - t, & \text{if } 1 \le t \le 2, \\ 0, & \text{if } 2 < t. \end{cases}$$

MAT 271E – Homework 6

Due 15.04.2015

1. You and your friend play a game as follows. You toss a fair coin until the first Head occurs and your friend rolls a fair die for each toss. Save for this stopping condition, assume that the tosses and the rolls are otherwise independent. What is the expected sum of the die rolls?

   Solution. In the following, we denote the sum of the die rolls as X.

   Let us define the events

   $$A_n = \{\text{the first Head occurs at the } n^{\text{th}} \text{ toss}\},$$

   for $n \geq 1$. Observe that A_n's form a partition of the sample space and $P(A_n) = (1/2)^n$.

   Note now that the expected value of the roll of a fair die is,

   $$\sum_{i=1}^{6} i \frac{1}{6} = \frac{7}{2}.$$

   If we roll the die n times, the expected value of the sum (under the assumption that the rolls are independent) is therefore $7n/2$. Thus,

   $$\mathbb{E}(X|A_n) = \frac{7}{2}n.$$

   We finally obtain,

   $$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{E}(X|A_n)\, P(A_n) = \sum_{n=1}^{\infty} \frac{7}{2}\, n\, \frac{1}{2^n} = 7.$$

2. Suppose (X_1, X_2) is uniformly distributed on the triangle formed by the points $(0,0)$, $(1,0)$, $(0,1)$. That is,

   $$f_{X_1,X_2}(t,u) = \begin{cases} 2, & \text{if } 0 \leq t,\ 0 \leq u, \text{ and } t + u \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the marginal pdfs of X_1 and X_2.
   (b) Determine if X_1 and X_2 are independent.
   (c) Compute $P(X_1 \leq t | X_2 = 1/2)$, $f_{X_1|X_2}(t|1/2)$ and $\mathbb{E}(X_1|X_2 = 1/2)$.
   (d) Compute $\mathbb{E}(X_1|X_1 \leq X_2)$.

   Solution. (a) We compute

   $$f_{X_1}(t) = \int_{-\infty}^{\infty} f_{X_1,X_2}(t,u)\, du = \begin{cases} 0, & \text{if } t \notin [0,1], \\ \int_0^{1-t} 2\, du = 2(1-t), & \text{if } t \in [0,1]. \end{cases}$$

   It similarly follows that

   $$f_{X_2}(u) = \begin{cases} 0, & \text{if } u \notin [0,1], \\ 2(1-u), & \text{if } u \in [0,1]. \end{cases}$$

   (b) Observe that

   $$f_{X_1}(t)\, f_{X_2}(u) \neq f_{X_1,X_2}(t,u)$$

   for all (t,u). Therefore X_1 and X_2 are not independent.

   (c) Note that

   $$f_{X_1|X_2}(t|u) = \frac{f_{X_1,X_2}(t,u)}{f_{X_2}(u)} = \begin{cases} 1/(1-u), & \text{if } t \geq 0,\ u \geq 0,\ t+u \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Thus,

   $$f_{X_1|X_2}(t|1/2) = \begin{cases} 2, & \text{if } 0 \leq t \leq 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

   We now obtain,

   $$P(X_1 \leq t | X_2 = 1/2) = \begin{cases} 0, & \text{if } t \leq 0, \\ 2t, & \text{if } 0 \leq t \leq 1/2, \\ 1, & \text{if } 1/2 < t. \end{cases}$$

   Also,

   $$\mathbb{E}(X_1|X_2 = 1/2) = \int_{-\infty}^{\infty} t\, f_{X_1|X_2}(t|1/2)\, dt = \frac{1}{4}.$$

(d) Let A be the event that $X_1 \le X_2$. To compute $\mathbb{E}(X_1|A)$, let us first find the conditional joint pdf $f_{X_1,X_2|A}(t,u)$. Notice that since X_1 and X_2 is uniformly distributed on the triangle Δ between the points $(0,0),(1,0),(0,1)$, given A, the pair is uniformly distributed on $\Delta \cap A$ (why?). Therefore, the conditional joint pdf is given as,

$$f_{X_1,X_2|A}(t,u) = \begin{cases} 4, & \text{if } 0 \le t,\ 0 \le u, \text{ and } t+u \le 1, t \le u, \\ 0, & \text{otherwise.} \end{cases}$$

We can now compute $\mathbb{E}(X_1|A)$ using this joint conditional pdf,

$$\begin{aligned} \mathbb{E}(X_1|A) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\, f_{X_1,X_2|A}(t,u)\, du\, dt \\ &= \int_0^{1/2}\int_t^{1-t} t\, 4\, du\, dt \\ &= \int_0^{1/2}\int_t^{1-t} t\, 4\,(1-2t)\, dt \\ &= \frac{1}{6}. \end{aligned}$$

3. Suppose X and Y are independent standard normal random variables. Also, let $Z = X + 2Y$.

(a) Find the joint pdf of X and Z, that is $f_{X,Z}(t,u)$.

(b) Find the pdf of Z.

(c) Compute $\mathbb{E}(X|Z=u)$.

Solution. (a) Notice that given $X = t$, Z can be written as $Z = t + 2Y$. But we know that if $U = aV + b$, then

$$f_U(u) = \frac{1}{|a|} f_V\left(\frac{u-b}{a}\right).$$

Since Y is standard normal, it therefore follows that

$$f_{Z|X}(u|t) = \frac{1}{2} f_Y\left(\frac{u-t}{2}\right) = \frac{1}{\sqrt{2\pi}\,2}\exp\left(-\frac{1}{2}\left(\frac{u-t}{2}\right)^2\right).$$

Notice that this is the pdf of a Gaussian with mean t, variance 4. Multiplying with $f_X(t)$, we obtain the joint pdf as,

$$f_{Z,X}(u,t) = \frac{1}{4\pi}\exp\left(-\frac{1}{8}(u-t)^2 - \frac{1}{2}t^2\right).$$

(b) We can find the pdf of Z in different ways. First, we can use the joint pdf of Z and X to find the marginal pdf as,

$$f_Z(t) = \int_{-\infty}^{\infty} f_{Z,X}(t,u)\, dt.$$

For that, notice that (check this!)

$$f_{Z,X}(u,t) = \frac{1}{\sqrt{2\pi}\sqrt{5}}\exp\left(-\frac{1}{10}u^2\right) \times \frac{1}{\sqrt{2\pi}\,2/\sqrt{5}}\exp\left(-\frac{5}{8}(t-u/5)^2\right).$$

We recognize the second term as the pdf of a Gaussian random variable centered around u with variance $4/5$. Thus it integrates to 1. Using this observation, we obtain,

$$\begin{aligned} f_Z(t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{5}}\exp\left(-\frac{1}{10}u^2\right) \times \frac{1}{\sqrt{2\pi}\,2/\sqrt{5}}\exp\left(-\frac{5}{8}(t-u/5)^2\right) dt \\ &= \frac{1}{\sqrt{2\pi}\sqrt{5}}\exp\left(-\frac{1}{10}u^2\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,2/\sqrt{5}}\exp\left(-\frac{5}{8}(t-u/5)^2\right) du \\ &= \frac{1}{\sqrt{2\pi}\sqrt{5}}\exp\left(-\frac{1}{10}u^2\right). \end{aligned}$$

Thus Z is Gaussian with mean 0, variance 5.

The second approach is arguably simpler. For that, remember that the sum of Gaussian random variables is Gaussian and only two parameters (mean and variance) are necessary to describe a Gaussian random variable. For our case, we know X and Y are independent and zero-mean. Therefore Z is also zero mean and has variance $1 + 4 = 5$.

(c) Note that

$$f_{X|Z}(t|u) = \frac{f_{Z,X}(u,t)}{f_Z(u)} = \frac{1}{\sqrt{2\pi}\,2/\sqrt{5}}\exp\left(-\frac{5}{8}(t-u/5)^2\right).$$

Thus, given $Z = u$, X is a Gaussian random variable with mean $u/5$, variance $4/5$. Therefore, $\mathbb{E}(X|Z=u) = u/5$.

MAT 271E – Homework 7

Due 06.05.2015

1. Let X be a random variable, uniformly distributed on $[0, 1]$. Also, let Z be another random variable defined as $Z = X^\alpha$, where α is an non-zero but unknown constant.

   (a) Compute $f_Z(t)$, the pdf of Z, in terms of α.

   (b) Suppose we are given two independent realizations of Z as z_1, z_2. Find the maximum likelihood (ML) estimate of α, in terms of z_1, z_2.

   (c) Evaluate the ML estimate you found in part (b) for $z_1 = e^{-3}$, $z_2 = e^{-4}$.

   Solution. (a) Notice that

   $$F_Z(t) = P(Z \le t) = P(X^\alpha \le t) = P(X \le t^{1/\alpha}) = \begin{cases} 0, & \text{if } t < 0, \\ t^{1/\alpha}, & \text{if } t \in [0, 1], \\ 1, & \text{if } 1 < t \end{cases}$$

   Differentiating wrt t, we obtain the pdf of Z.

   $$f_Z(t) = \begin{cases} \frac{1}{\alpha}\, t^{1/\alpha - 1}, & \text{if } t \in [0, 1], \\ 0, & \text{if } t \notin [0, 1]. \end{cases}$$

   (b) Given the observations, z_1, z_2, the likelihood function is,

   $$L(\alpha) = f_Z(z_1)\, f_Z(z_2) = \frac{1}{\alpha^2}\, (z_1\, z_2)^{1/\alpha}.$$

   Let $p = z_1\, z_2$ for simplicity of notation. To find the maximizer of $L(\alpha)$, namely $\hat{\alpha}$, we set the derivative to zero,

   $$L'(\hat{\alpha}) = -2\, \frac{1}{\hat{\alpha}^3} p^{1/\alpha - 1} + \frac{1}{\alpha^2}\, p^{1/\alpha - 1} \left(-\frac{p}{\alpha^2} \right) = 0.$$

   Solving this equation, we obtain the ML estimate as

   $$\hat{\alpha} = -\frac{\ln(z_1\, z_2)}{2}.$$

   (c) Plugging in the values for z_1 and z_2, the ML estimate is found for this case as,

   $$\hat{\alpha} = \frac{3 + 4}{2}.$$

2. Consider two independent random variables, X, Y, that are uniformly distributed on $[0, \theta]$. Using X and Y, we define a new random variable as $Z = \max(X, Y)$.

   (a) Suppose θ is an unknown parameter of interest. A student proposes to use Z as an estimator for θ. Is Z a biased or an unbiased estimator for θ? If it is biased, can you propose an unbiased estimator?

   (b) Biased or not, we decide to use Z as the estimator for θ. Find the value of c so that the interval $[Z, Z + c]$ contains θ with probability 99/100.

   Solution. (a) Let us first find the pdf of Z (it will facilitate our job in (b)). Notice that, because X and Y are independent,

   $$F_Z(t) = P(Z \le t) = P\Big((X \le t) \cap (Y \le t) \Big) = P(X \le t)\, P(Y \le t) = \begin{cases} 0, & \text{if } t < 0, \\ t^2/\theta^2, & \text{if } t \in [0, \theta], \\ 1, & \text{if } \theta < t. \end{cases}$$

   We obtain the pdf by differentiating as,

   $$f_Z(t) = \begin{cases} 2t/\theta^2, & \text{if } t \in [0, \theta], \\ 0, & \text{if } t \notin [0, \theta]. \end{cases}$$

   We can now compute $\mathbb{E}(Z)$ as,

   $$\mathbb{E}(Z) = \int_0^\theta t\, \frac{2t}{\theta^2}\, dt = \frac{2}{3}\theta.$$

   Since $\mathbb{E}(Z) \neq \theta$, Z is biased as an estimator of θ. However, we see that

   $$\hat{\theta} = \frac{3}{2}\, Z = \frac{3}{2}\, \max(X, Y)$$

   is an unbiased estimator of θ.

(b) Notice that

$$\begin{aligned}P(\theta \in [Z, Z+c]) &= P(Z \le \theta \le Z+c)\\&= P\big((Z \le \theta) \cap (\theta \le Z+c)\big)\\&= P\big((Z \le \theta) \cap (\theta - c \le Z)\big)\\&= P\big(\theta - c \le Z \le \theta\big)\\&= \int_{\theta-c}^{\theta} f_Z(t)\,dt\\&= 1 - \frac{(\theta-c)^2}{\theta^2}.\end{aligned}$$

Solving the equation,

$$1 - \frac{(\theta-c)^2}{\theta^2} = \frac{99}{100},$$

we obtain c as, $c = \frac{9\theta}{10}$.

3. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with pdf

$$f_{X_i}(t) = \begin{cases} 0, & \text{if } t < 0,\\ \ln(\theta)\,\theta^{-t}, & \text{if } t \ge 0,\end{cases}$$

where $\theta > 1$ is an unknown constant. Find the maximum likelihood estimator for θ in terms of $X_1, X_2, \ldots, X_n$. Specify whether the estimator you found is biased or not.

Solution. Since X_i's are independent, their joint pdf is given as,

$$f(t_1, t_2, \ldots, t_n) = \prod_{i=1}^{n} f_{X_i}(t_i) = \begin{cases} (\ln\theta)^n\,\theta^{-\sum_i t_i}, & \text{if all } t_i \ge 0,\\ 0, & \text{otherwise.}\end{cases}$$

Suppose we are given the realizations of X_i as x_i. Then, the likelihood function is obtained by evaluating the joint pdf at $t_i = x_i$, giving us,

$$L(\theta) = (\ln\theta)^n\,\theta^{-s}, \tag{1}$$

where $s = \sum_{i=1}^{n} x_i$. To find the maximizer of $L(\theta)$, we set the derivative with respect to θ to zero and obtain the equation,

$$L'(\hat\theta) = \frac{n}{\theta}\,(\ln\theta)^{n-1}\,\theta^{-s} + (\ln\theta)^n(-s)\,\theta^{-s-1}\Big|_{\theta=\hat\theta} = 0$$

Solving this equation, we obtain the ML estimate as,

$$\hat\theta = e^{n/s}.$$

This estimator is biased. In fact, for $n = 1$, the expected value of $\hat\theta$ is infinite.

4. Consider a disk with an unknown radius r. We are interested in the area of the disk. For this, we measure the radius n times but each measurement contains some error. Specifically, suppose that the measurements are of the form $X_i = r + Z_i$ for $i = 1, 2, \ldots, n$, where Z_i's are independent zero-mean Gaussian random variables with known variance σ^2. A professor suggests that we use

$$\hat A = \pi\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)$$

as an estimator of the area. Determine if $\hat A$ is biased or not. If it is biased, propose an unbiased estimator for the area of the disk.

Solution. Notice that

$$\begin{aligned}\mathbb{E}(\hat A) &= \mathbb{E}\left(\frac{\pi}{n}\sum_{i=1}^{n} X_i^2\right)\\&= \frac{\pi}{n}\sum_{i=1}^{n}\mathbb{E}(X_i^2)\\&= \frac{\pi}{n}\sum_{i=1}^{n}\mathbb{E}(r^2 + 2r\,Z_i + Z_i^2)\\&= \frac{\pi}{n}\sum_{i=1}^{n} r^2 + \sigma^2\\&= \pi\,r^2 + \pi\,\sigma^2.\end{aligned}$$

Thus, $\hat A$ is biased. Observe however that an unbiased estimator can be derived easily from $\hat A$ as,

$$\bar A = \hat A - \pi\,\sigma^2 = \pi\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) - \sigma^2\right]$$

MAT 271E – Probability and Statistics

Midterm Examination I

11.03.2015

Student Name : ______________________

Student Num. : ______________________

4 Questions, 120 Minutes

Please Show Your Work for Full Credit!

(25 pts) 1. An urn contains one red, two green, and three blue balls. A ball is drawn randomly from the urn. Then without putting the drawn ball back, another ball is drawn randomly.

(a) Propose a sample space for this experiment.

(b) Compute the probability that the second ball is blue.

(c) Given that the second ball is blue, compute the probability that the first ball is red.

(25 pts) 2. Suppose that a coin is tossed five times. The coin is biased and $P(\text{Head}) = p$. Assume that the tosses are independent.

(a) Consider the event $A = \{\text{all tosses are Tails}\}$. Compute $P(A)$, the probability of A.

(b) Consider the event $B = \{\text{at least one Head occurs}\}$. Compute $P(B)$.
(Hint : Note that 'at least one' means 'one or more than one'. Think of how A and B are related.)

(c) Consider the event $C = \{\text{at least one Tail occurs}\}$. Given the event B in (b), compute $P(C|B)$, the conditional probability of C given B.

(25 pts) 3. Suppose X is a random variable whose probability mass function (PMF) is given as

$$P_X(k) = \begin{cases} c\,2^{-k}, & \text{if } k \text{ is an integer in } [-2, 2], \\ 0, & \text{otherwise,} \end{cases}$$

where c is a constant. Suppose also that $Y = |X|$.

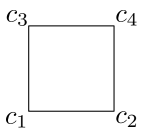
(a) Determine c.

(b) Compute the probability of the event $\{X \leq 1\}$ (give your answer in terms of c if you have no answer for part (a)).

(c) Compute the probability of the event $\{Y \leq 1\}$.

(d) Find P_Y, the PMF of Y.

(25 pts) 4. Consider a square whose corners are labeled as c_i (see below). A particle is placed on one of the corners and starts moving from one corner to another connected by an edge at each step. Notice that each corner is connected to only two corners.



If the particle reaches c_4, it is trapped and stops moving. Assume that the steps taken by the particle are independent and the particle chooses its next stop randomly (i.e., all possible choices are equally likely). Suppose that the particle is initially placed at c_1. Also, let X denote the total number of steps taken by the particle to reach c_4.

(a) Find the probability that $X = 1$.

(b) Find the probability that $X = 2$.

(c) Find the probability that $X = 4$.

(d) Write down P_X, the probability mass function (PMF) of X.

(e) Compute $\mathbb{E}(X)$, the expected value of X.

MAT 271E – Probability and Statistics

Midterm Examination II

22.04.2015

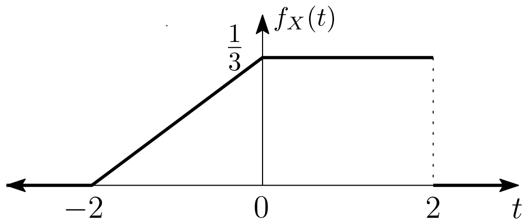
Student Name : ______________________

Student Num. : ______________________

4 Questions, 100 Minutes

Please Show Your Work for Full Credit!

(25 pts) 1. Consider a random variable X whose probability density function (pdf) is as shown below.



Let us define the events A and B as $A = \{X > 0\}$, $B = \{|X| > 1\}$.

(a) Compute the probability of A.

(b) Compute the probability of B.

(c) Compute the conditional probability $P(A|B)$.

(d) Compute $\mathbb{E}(X)$.

(25 pts) 2. Suppose X is a <u>continuous</u> random variable with cumulative distribution function (cdf) $\Phi(t) = P(X < t)$, and Z is a Bernoulli random variable with probability mass function (PMF),

$$P_Z(k) = \begin{cases} 1/2, & \text{if } k = 0, \\ 1/2, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also, let $Y = X + Z$. Assume that X and Z are independent.

(a) Find an expression for $P(Y > 0)$ in terms of $\Phi(t)$.
(Note: Pay attention to the direction of the inequality).

(b) Find an expression for $F_Y(t) = P(Y \leq t)$ in terms of $\Phi(t)$.

(25 pts) 3. Suppose X is uniformly distributed on the interval $[-2, 2]$, that is,

$$f_X(t) = \begin{cases} 1/4, & \text{if } -2 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Also, let $Z = X^2 - 3$.

(a) Find $f_Z(t)$, the probability density function (pdf) of Z.

(b) Compute the probability that $X^2 - 2X > 3$.

(25 pts) 4. Suppose we roll a fair die until we observe a 6. Assume that the tosses are independent. Let X be the total sum of the rolls.

(a) Find the probability that we roll n times, where $n \geq 1$.

(b) Ler A_2 be the event that we roll twice. Compute $\mathbb{E}(X|A_2)$.

(c) Compute $\mathbb{E}(X)$.

MAT 271E – Probability and Statistics

Final Examination

21.05.2015

Student Name : ______________________

Student Num. : ______________________

5 Questions, 120 Minutes

Please Show Your Work for Full Credit!

(20 pts) 1. There are 4 white and 6 black balls in an urn (10 balls in total). Your friend starts drawing balls randomly, without replacement, and stops drawing when he draws a ball that has the same color as the first ball (note that, at least two balls have to be drawn and after the k^{th} draw, there are $10-k$ balls in the urn).

(a) Find the probability that your friend draws two balls and stops.

(b) Find the probability that your friend draws three balls and stops.

(c) Given that your friend stops right after the third draw, compute the conditional probability that the first ball is white.

(20 pts) 2. Suppose X, Y are independent discrete random variables whose probability mass functions (PMF) are given as

$$P_X(k) = \begin{cases} \frac{1}{6}, & \text{if } k = 0, \\ \frac{2}{6}, & \text{if } k = 1, \\ \frac{3}{6}, & \text{if } k = 2, \\ 0, & \text{otherwise,} \end{cases} \qquad P_Y(k) = \begin{cases} \frac{1}{2}, & \text{if } k = 0, \\ \frac{1}{2}, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also, let the random variable Z be defined as $Z = X + Y$.

(a) Compute $\mathbb{E}(X)$, the expected value of X.

(b) Compute $\mathbb{E}(X^2)$, the expected value of X^2.

(c) Compute the probabilities of the events $\{Z = 0\}$ and $\{Z = 1\}$.

(d) Find P_Z, the PMF of Z.

(20 pts) 3. Suppose X is a continuous random variable whose probability density function (pdf) is given as

$$f_X(t) = \begin{cases} t/2, & \text{if } 0 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Also, let $Y = -X + 1$.

(a) Compute $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{E}(X\,Y)$.

(b) Are X and Y independent or not? Briefly explain.

(c) Compute the probability of the event $\{Y \leq 0\}$.

(d) Find $f_Y(t)$, the pdf of Y.

(20 pts) 4. Suppose X and Y are independent continuous random variables whose pdfs are given as

$$f_X(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{otherwise,} \end{cases} \qquad f_Y(t) = \begin{cases} 2 - 2t, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

(a) Compute $P(Y \leq s)$, for $s \in [0, 1]$.

(b) Compute $P(Y \leq X)$.

(c) Compute $P(Y \leq X^2)$.

(20 pts) 5. Suppose X is a continuous random variable uniformly distributed on $[-1, 1]$. Note that the pdf of X is given by

$$f_X(t) = \begin{cases} 1/2, & \text{if } -1 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also, let $Y = \theta X$, where θ is an unknown non-negative constant (so that $\theta = |\theta|$).

(a) Compute $\mathbb{E}(Y^2)$, and $\mathbb{E}(|Y|)$, possibly in terms of the unknown θ.

(b) Find some 'c' (possibly in terms of θ) such that $P\{|Y| - c \leq \theta \leq |Y| + c\} = 1/3$.

(c) Find an unbiased estimator for θ in terms of Y.