MAT 271E – Probability and Statistics

Spring 2016

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Class Meets :	13.30 – 16.30, Wednesday EEB 2106
Office Hours :	10.00 – 12.00, Monday
Textbook :	D. B. Bertsekas, J. N. Tsitsiklis, 'Introduction to Probability', 2^{nd} Edition, Athena-Scientific.
Grading :	Homeworks (10%), 2 Midterms (25% each), Final (40%).
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Tentative Course Outline

- Probability Space
  Probability models, conditioning, Bayes' rule, independence.
- Discrete Random Variables
  Probability mass function, functions of random variables, expectation, joint PMFs, conditioning, independence.
- General Random Variables
  Probability distribution function, cumulative distribution function, continuous Bayes' rule, correlation, conditional expectation.
- Limit Theorems
  Law of large numbers, central limit theorem.
- Introduction to Statistics
  Parameter estimation, linear regression, hypothesis testing.

MAT 271E – Homework 1

Due 24.02.2016

1. You have a regular, unbiased coin and a regular, unbiased die. Consider the following two step experiment.

   (i) You toss the coin.

   (ii) • If the toss is a 'Head', you toss the coin again.
   • If the toss is a 'Tail', you roll the die.

   Propose a sample space for this experiment.

   Solution. Below is one reasonable sample space for this experiment. Here, H represents a 'Head' and T a 'Tail'.

   $$\Omega = \{(H,H),\,(H,T),\,(T,1),\,(T,2),\,(T,3),\,(T,4),\,(T,5),\,(T,6)\}.$$

2. You have a regular, unbiased coin. You start tossing the coin. You stop if

   - you observe two Heads before the fourth toss,
     or,
   - you tossed the coin four times.

   (a) Propose a sample space for this experiment.

   (b) For the experiment described above, let us define the events

   $$\begin{aligned} B_k &= \{\text{a Head occurs at the } k^{\text{th}} \text{ toss}\}, \text{ for } k = 1,2,3,4, \\ A &= \{\text{you toss the coin three times and stop}\}. \end{aligned}$$

   Express the event A in terms of the events B_k, for $k = 1,2,3,4$.
   (You can use as many B_k's as you like).

   Solution. (a) A minimal sample space for this experiment is

   $$\begin{aligned} \Omega = \{&(H,H),\,(T,H,H),\,(H,T,H),\,(H,T,T,H),\,(H,T,T,T), \\ &(T,H,T,H),\,(T,H,T,T),\,(T,T,H,H),\,(T,T,H,T),\,(T,T,T,H),\,(T,T,T,T)\} \end{aligned}$$

   (b) Notice that A occurs if and only if $\{(T,H,H),(H,T,H)\}$ occurs. Therefore,

   $$A = (B_1^c \cap B_2 \cap B_3) \cup (B_1 \cap B_2^c \cap B_3).$$

3. You have four coins in your pocket. One of the coins has 'Heads' on both faces. The other three are regular, unbiased coins. You randomly pick one of the coins, toss it and observe a 'Head'. Answer the following questions by taking into account this information.

   (a) What is the probability that the other face is also a 'Head'?

   (b) Without looking at the other face, you randomly pick one of the remaining coins. What is the probability that this second coin has 'Heads' on both faces?

   Solution. (a) Let us define the events

   $$\begin{aligned} B &= \{\text{the randomly picked coin has heads on both faces}\}, \\ H &= \{\text{we observe a head after the toss}\}. \end{aligned}$$

   We are asked to compute $P(B|H)$. We compute this as,

   $$\begin{aligned} P(B|H) &= \frac{P(H|B)\,P(B)}{P(H)} \\ &= \frac{P(H|B)\,P(B)}{P(H|B)\,P(B) + PH|B^c\,P(B^c)} \\ &= \frac{1\cdot(1/4)}{1\cdot(1/4)+(1/2)\cdot(3/4)} = \frac{2}{5}. \end{aligned}$$

(b) Let us define the event S, in addition to H and B as,

$$S = \{\text{the second randomly picked coin has heads on both faces}\}.$$

Notice that we need to compute $P(S|H)$. Note also that $P(H) = 5/8$ from part (a). We now have,

$$\begin{aligned} P(S|H) &= \frac{P(H|S)\,P(S)}{P(H)} \\ &= \frac{1/2\,P(S)}{5/8}. \end{aligned}$$

We can compute $P(S)$ by conditioning on B. That is,

$$\begin{aligned} P(S) &= P(S|B)\,P(B) + P(S|B^c)\,P(B^c) \\ &= 0 \cdot (1/4) + (1/3) \cdot (3/4) = 1/4. \end{aligned}$$

Thus, $P(S|H) = 1/5$.

4. Two specialists claim that they can tell whether a painting is original or a reproduction by looking at the painting. However it is known from past experience that each specialist makes mistakes with probability p. (Note that there are two types of mistakes – (i) the painting is original but the specialist decides it is a reproduction, (ii) the painting is a reproduction but the specialist decides it is original. The probability of both types of mistakes are the same and equal to p).

Suppose it is known that there are 3 reproductions and 7 originals in a collection of 10 paintings but this information is not provided to the specialists. A painting is selected randomly from the collection and is presented to the specialists. Assume that the specialists decide on their own (they do not know what the other decides).

(a) Find the probability that the first specialist decides that the given painting is original.

(b) Given that the first specialist decides the painting is original, find the conditional probability that the selected painting is actually a reproduction.

(c) Given that the first specialist decides the painting is original, find the conditional probability that the second painter decides the painting is original.

Solution. (a) We are asked to compute the probability of the event

$$D_1 = \{\text{the first specialist decides that the given painting is an original}\}.$$

We will compute this probability by conditioning on the following event

$$O = \{\text{the selected painting is an original}\}.$$

We find,

$$P(D_1) = P(D_1|O)\,P(O) + P(D_1|O^c)\,P(O^c) = (1-p) \cdot (7/10) + p \cdot (3/10) = \frac{7}{10} - p\frac{4}{10}.$$

(b) We are asked to compute $P(O^c|D_1)$. Observe that

$$P(O^c|D_1) = \frac{P(D_1|O^c)\,P(O^c)}{P(D_1)} = \frac{p \cdot (3/10)}{(7-4p)/10} = \frac{3p}{7-4p}.$$

(c) Let us define the event

$$D_2 = \{\text{the second specialist decides that the given painting is an original}\}.$$

We are asked to compute $P(D_2|D_1)$. Note that

$$P(D_2|D_1) = \frac{P(D_1 \cap D_2)}{P(D_1)}.$$

We already computed $P(D_1)$. To compute $P(D_1 \cap D_2)$, let us condition on O.

$$P(D_1 \cap D_2) = P(D_1 \cap D_2|O)\,P(O) + P(D_1 \cap D_2|O^c)\,P(O^c) = (1-p)^2 \cdot (7/10) + p^2 \cdot (3/10).$$

Thus we have,

$$P(D_2|D_1) = \frac{7 - 14p + 10p^2}{7 - 4p}$$

5. An urn initially contains 5 white balls and 1 black ball. Your friend removes 2 balls from the urn randomly so that there are now a total of 4 balls.

   (a) What is the probability that the black ball is still in the urn?

   (b) Suppose you pick randomly one of the remaining balls in the urn. What is the probability that the ball you pick is black?

   (Hint : Think in terms of events. You do not need sophisticated counting tricks.)

   Solution. (a) We need to compute the probability of the event

   $$B = \{\text{the black ball is still in the urn after the removal of two balls}\}.$$

   Let us define the events

   $$W_1 = \{\text{the first removal is white}\},$$
   $$W_2 = \{\text{the second removal is white}\}.$$

   Notice that $B = W_1 \cap W_2$. Therefore,

   $$P(B) = P(W_1 \cap W_2) = P(W_2|W_1) \cdot P(W_1) = (4/5) \cdot (5/6) = 2/3.$$

   (b) We are asked to compute the probability of the event

   $$C = \{\text{the ball you pick is black}\}.$$

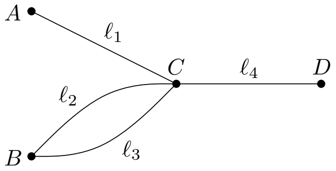
   This is easier to compute if we condition on B.

   $$P(C) = P(C|B) \cdot P(B) + P(C|B^c) \cdot P(B^c) = (1/4) \cdot (2/3) + 0 \cdot (1/3) = 1/6.$$

MAT 271E – Homework 2

Due 02.03.2016

1. Consider a network with four nodes (A, B, C, D) and four links (ℓ_1, ℓ_2, ℓ_3, ℓ_4) as shown below.



Assume that at a particular time, a particular link is operational with probability 'p'. Assume also that the conditions of the links (i.e. whether they are operational or not) are independent of each other. We say that two nodes can communicate if there exists at least one path with operational links between them.

(a) Compute the probability that A can communicate with D.

(b) Compute the probability that B can communicate with C.

(c) Compute the probability that A and B can both communicate with C.

(d) Compute the probability that A and B can both communicate with D.

Solution. Let us define the events

$$E_k = \{\ell_k \text{ is operational}\}, \text{ for } k = 1, 2, \ldots, 5,$$

to be used in the sequel. We are given that E_k are independent and $P(E_k) = p$.

(a) Define the event,

$$E_{AD} = \{A \text{ can communicate with } D\}.$$

We are asked to find $P(E_{AD})$. Observe that $E_{AD} = E_1 \cap E_4$. By the independence of E_k's, we have $P(E_{AD}) = P(E_1)\,P(E_4) = p^2$.

(b) Define the event,

$$E_{BC} = \{B \text{ can communicate with } C\}.$$

Observe that $E_{BC} = E_2 \cup E_3$. Note that $P(E_2 \cup E_3) = P(E_2) + P(E_3) - P(E_2 \cap E_3) = 2p - p^2$.

(c) Define the event

$$E_{AC} = \{A \text{ can communicate with } C\}.$$

We are asked to compute $P(E_{AC} \cap E_{BC})$. Notice that $E_{AC} \cap E_{BC} = E_1 \cap (E_2 cap E_3)$. Again, by the independence of E_1 from E_2, E_3, we find that $P(E_{AC} \cap E_{BC}) = p\,(2p - p^2)$.

(d) Define the events

$$E_{CD} = \{C \text{ can communicate with } D\},$$
$$E_{AD} = \{A \text{ can communicate with } D\},$$
$$E_{BD} = \{B \text{ can communicate with } D\}.$$

Observe that $E_{AD} \cap E_{BD} = E_{AC} \cap E_{BC} \cap E_{CD}$. Observe that $E_{CD} = E_4$. Thanks to independence, we find $P(E_{AD} \cap E_{BD}) = P(E_{AC} \cap E_{BC})\,P(E_{CD}) = p^2\,(2p - p^2)$.

2. Suppose that a coin is tossed five times. The coin is biased and $P(\text{Head}) = p$. Assume that the tosses are independent.

(a) Consider the event $A = \{\text{all tosses are Tails}\}$. Compute $P(A)$, the probability of A.

(b) Consider the event $B = \{\text{at least one Head occurs}\}$. Compute $P(B)$.
(Hint : Note that 'at least one' means 'one or more than one'. Think of how A and B are related.)

(c) Consider the event $C = \{\text{at least one Tail occurs}\}$. Given the event B in (b), compute $P(C|B)$, the conditional probability of C given B.

Solution. Let us define the events

$$E_k = \{k^{\text{th}} \text{ toss is a Head}\}, \text{ for } k = 1, 2, \ldots, 5.$$

By assumption, E_k's are independent.

(a) Notice that $A = E_1^c \cap E_2^c \cap \cdots \cap E_5^c$. Thanks to independence, we find $P(E) = \prod_{k=1}^5 P(E_k) = (1-p)^5$.

(b) Notice that $B = A^c$. Therefore, $P(B) = 1 - (1-p)^5$.

(c) Notice that $B \cap C = \{\text{there is at least one Tail and at least one Head}\}$. The complement of this event is, $(B \cap C)^c = \{HHHHH, TTTTT\}$. Therefore, $P(B \cap C) = 1 - P\big((B \cap C)^c\big) = 1 - p^5 - (1-p)^5$. So,

$$P(C|B) = \frac{P(B \cap C)}{P(B)} = \frac{1 - p^5 - (1-p)^5}{1 - (1-p)^5}.$$

3. You toss an unbiased coin until you observe two 'Heads' (in total) – once you observe two 'Heads', you stop tossing. Assume that each toss is independent of the previous tosses.

(a) What is the probability that you stop after two tosses?

(b) What is the probability that you stop after three tosses?

(c) Let X be the number of tosses. Write down the probability mass function (PMF) of X.

Solution. Let us define the events,

$$E_k = \{\text{you stop after } k \text{ tosses}\}, \text{ for } k \geq 2.$$

(a) Notice that $E_2 = \{H\,H\}$. Since the coin is unbiased, $P(H) = 1/2$. Therefore, by independence, $P(E_2) = (1/2)^2$.

(b) Notice that $E_3 = \{HTH, THH\}$. Therefore, $P(E_3) = 2\,(1/2)^3$.

(c) Observe that the event $\{X = k\}$ is equivalent to E_k. Notice that E_k contains sequences of H, T that end with H and that contain a single H in the remaining $k-1$ entries. There are $k-1$ such sequences and the probability of each sequence is $(1/2)^k$. Therefore, $P(E_k) = (k-1)\,(1/2)^k$. Thus the PMF of X is given as,

$$P_X(k) = P(X = k) = \begin{cases} (k-1)\,(1/2)^k, & \text{if } k \text{ is an integer greater than } 1, \\ 0, & \text{otherwise.} \end{cases}$$

4. Suppose X is a random variable whose probability mass function (PMF) is given as

$$P_X(k) = \begin{cases} c\,2^{-k}, & \text{if } k \text{ is an integer in } [-2, 2], \\ 0, & \text{otherwise,} \end{cases}$$

where c is a constant. Suppose also that $Y = |X|$.

(a) Determine c.

(b) Compute the probability of the event $\{X \leq 1\}$ (give your answer in terms of c if you have no answer for part (a)).

(c) Compute the probability of the event $\{Y \leq 1\}$.

(d) Find P_Y, the PMF of Y.

Solution. (a) Recall that P_X sums to one. Therefore, we must have,

$$\sum_{k=-2}^{2} c\,2^{-k} = c\,\frac{31}{4} = 1 \quad \Longrightarrow \quad c = \frac{4}{31}.$$

(b) Notice that $\{X \leq 1\} = \{X = -2\} \cup \{X = -1\} \cup \{X = 0\} \cup \{X = 1\}$. Thus,

$$P(X \leq 1) = \sum_{k=-2}^{1} c\,2^{-k} = \frac{30}{31}.$$

(c) Notice that $\{Y \leq 1\} = \{X = -1\} \cup \{X = 0\} \cup \{X = 1\}$. Thus,

$$P(Y \leq 1) = \sum_{k=-1}^{1} c\,2^{-k} = \frac{14}{31}.$$

(d) Observe that $\{Y = 1\} = \{X = 1\} \cup \{X = -1\}$, $\{Y = 2\} = \{X = 2\} \cup \{X = -2\}$, $\{Y = 0\} = \{X = 0\}$. Therefore,

$$P(Y = k) = \begin{cases} P_X(0) = (4/31), & \text{if } k = 0, \\ P_X(-1) + P_X(1) = (10/31), & \text{if } k = 1, \\ P_X(-2) + P_X(2) = (17/31), & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

MAT 271E – Homework 3

Due 09.03.2016

1. There are 10 balls in an urn, numbered from 1 to 10. You draw a ball and then without putting it back, draw another ball. Let X be the number on the first ball. Let Y be the number on the second ball.

   (a) Find $P_X(k)$, the probability mass function (PMF) of X.

   (b) Find $P_Y(k)$, the PMF of Y.

   (c) Let us define a new random variable $Z = X + Y$. Find the expected value of Z.

   (Hint : Recall that $\displaystyle\sum_{n=1}^{k} n = \frac{k\,(k+1)}{2}$.)

   Solution. (a) Notice that any number is equally likely. Therefore the PMF of X is

$$P_X(k) = \begin{cases} \frac{1}{10}, & \text{if } k \in \{1, 2, \dots, 10\}, \\ 0, & \text{otherwise.} \end{cases}$$

   (b) By symmetry we can say that $P(Y = k) = P(Y = j)$ where k and j are two integers in the range $[1, 10]$. Therefore $P_Y(k) = P_X(k)$.
   For a more formal argument, we can condition on X. That is, if k is an integer in the range $[1, 10]$, we have

$$P(Y = k) = P(Y = k|X = k)\,P(X = k) + P(Y = k|X \neq k)\,P(X \neq k) = 0\,\frac{1}{10} + \frac{1}{9}\,\frac{9}{10} = \frac{1}{10}.$$

   Thus, $P_Y(k) = P_X(k)$.

   (c) Note that $\mathbb{E}(Z) = \mathbb{E}(X) + \mathbb{E}(Y)$. First,

$$\mathbb{E}(X) = \sum_{n=1}^{10} n\frac{1}{10} = \frac{10 \cdot 11}{2 \cdot 10} = \frac{11}{2}.$$

   Since $P_X(k) = P_Y(k)$, we have $\mathbb{E}(X) = \mathbb{E}(Y)$. Therefore, $\mathbb{E}(Z) = 11$.

2. Suppose that the PMF of a random variable X is given as

$$P_X(k) = \begin{cases} \left(\frac{1}{3}\right)^{k-1} \frac{2}{3}, & \text{if } k \text{ is a positive integer,} \\ 0, & \text{otherwise.} \end{cases}$$

   Also, let Y be another random variable defined as $Y = 2^X$.

   (a) Find the PMF of Y.

   (b) Find $\mathbb{E}(Y)$, the expected value of Y.
   (Note : Recall that '$\sum_{n=1}^{\infty} p^n = p/(1-p)$' if $|p| < 1$.)

   Solution. (a) Notice that $P(X = k) = P(Y = 2^k)$. Therefore,

$$P_Y(n) = \begin{cases} \left(\frac{1}{3}\right)^{\log_2(n)-1} \frac{2}{3}, & \text{if } \log_2(n) \text{ is a positive integer,} \\ 0, & \text{otherwise.} \end{cases}$$

   (b) We compute $\mathbb{E}(Y)$ as,

$$\mathbb{E}(Y) = \sum_{k=1}^{\infty} 2^k \left(\frac{1}{3}\right)^{k-1} \frac{2}{3} = \frac{4}{3} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = 4.$$

3. A fair die is rolled once. Based on this experiment, two random variables are defined as,

$$X = \begin{cases} 0, & \text{if the outcome is even,} \\ 1, & \text{if the outcome is odd,} \end{cases}$$

$$Y = \begin{cases} 0, & \text{if the outcome} \leq 3, \\ 1, & \text{if the outcome} > 3. \end{cases}$$

   Also, a random variable, Z, is defined as $Z = 2\,X - Y$.

(a) Write down the joint probability mass function (PMF) of X and Y (in the form of a table, if you like).

(b) Write down the PMF of Z.

(c) Find $\mathbb{E}(Z)$, the expected value of Z.

Solution. (a) Notice that for this experiment, the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. The event $(X = 0, Y = 0)$ is equivalent to $\{2\}$. Therefore $P_{X,Y}(0, 0) = 1/6$. The event $(X = 0, Y = 1)$ is equivalent to $\{4, 6\}$, so $P_{X,Y}(0, 1) = 2/6$. The event $(X = 1, Y = 0)$ is equivalent to $\{1, 3\}$, so $P_{X,Y}(1, 0) = 2/6$. Finally, the event $(X = 1, Y = 1)$ is equivalent to $\{5\}$, so$P_{X,Y}(1, 1) = 1/6$. $P_{X,Y}(x, y) = 0$ for any other (x, y) pair.

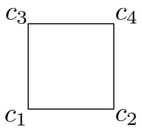
(b) Z can take at most four different values with non-zero probability. Those are, 0 (when $X = 0, Y = 0$); -1 (when $X = 0, Y = 1$), 2 (when $X = 1, Y = 0$); 1 (when $X = 1, Y = 1$). Therefore,

$$P_Z(k) = \begin{cases} P_{X,Y}(0,1) = 2/6, & \text{if } k = -1, \\ P_{X,Y}(0,0) = 1/6, & \text{if } k = 0, \\ P_{X,Y}(1,1) = 1/6, & \text{if } k = 1, \\ P_{X,Y}(1,0) = 2/6, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

(c) We can compute $\mathbb{E}(Z)$ using P_Z as,

$$\mathbb{E}(Z) = -1\,\frac{2}{6} + 0\,\frac{1}{6} + 1\,\frac{1}{6} + 2\frac{2}{6} = \frac{3}{6}.$$

4. Consider a square whose corners are labeled as c_i (see below).



A particle is placed on one of the corners and starts moving from one corner to another connected by an edge at each step. Notice that each corner is connected to only two corners. If the particle reaches c_4, it is trapped and stops moving. Assume that the steps taken by the particle are independent and the particle chooses its next stop randomly (i.e., all possible choices are equally likely). Suppose that the particle is initially placed at c_1. Also, let X denote the total number of steps taken by the particle to reach c_4.

(a) Find the probability that $X = 1$.

(b) Find the probability that $X = 2$.

(c) Find the probability that $X = 4$.

(d) Write down P_X, the probability mass function (PMF) of X.

(e) Compute $\mathbb{E}(X)$, the expected value of X.

Solution. A reasonable sample space for this experiment consists of sequences of c_i that end with c_4, where the only c_4 occurs at the end. In the following, let A_k denote the event $X = k$.

(a) The particle needs at least two steps, therefore $A_1 = \emptyset$ and $P(X = 1) = 0$.

(b) Notice that $A_2 = \{c_2c_4, c_3c_4\}$. The probability of any element of A_2 is equal to $1/4$. Therefore $P(A_2) = 1/2$.

(c) Notice that $A_4 = \{c_2c_1c_2c_4, c_2c_1c_3c_4, c_3c_1c_2c_4, c_3c_1c_3c_4\}$. Each sequence in A_4 can be formed by adding in front c_2c_1 or c_3c_1. Thus the number of sequences is doubled compared to A_2 but the probability of each sequence is now $1/2^4$. Therefore, $P(A_4) = 2^{4/2}\,\dfrac{1}{2^4} = \dfrac{1}{2^{4/2}}$.

(d) Notice that the pattern observed in part (c) holds throughout all k. That is, if k is even, then $P(A_k) = 2^{k/2}\,\dfrac{1}{2^k} = \dfrac{1}{2^{k/2}}$. If k is odd, then $P(A_k) = 0$. Therefore,

$$P_X(k) = \begin{cases} 2^{-k/2}, & \text{if } k \text{ is a positive even integer,} \\ 0, & \text{otherwise.} \end{cases}$$

(e) We compute $\mathbb{E}(X)$ using the definition as,

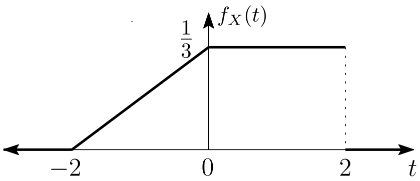
$$\mathbb{E}(X) = \sum_{\substack{k>0,\\ k \text{ even}}} k\,2^{-k/2} = \sum_{n=1}^{\infty} 2n\,2^{-n} = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1} = \frac{1}{(1-1/2)^2} = 4.$$

Note: To evaluate the series, consider differentiating both sides of $\sum_{k\geq 0} p^k = \dfrac{1}{1-p}$, for $|p| < 1$.

MAT 271E – Homework 4

Due 30.03.2016

1. Consider a random variable X whose probability density function (pdf) is as shown below.



Let us define the events A and B as $A = \{X > 0\}$, $B = \{|X| > 1\}$.

(a) Compute the probability of A.
(b) Compute the probability of B.
(c) Compute the conditional probability $P(A|B)$.
(d) Compute $\mathbb{E}(X)$.

Solution. (a) Notice that

$$P(X > 0) = \int_0^\infty f_X(t)\,dt = \int_0^2 \frac{1}{3}\,dt = \frac{2}{3}.$$

(b) Notice that

$$P(|X| > 1) = P(X < -1) + P(X > 1) = \int_{-\infty}^{-1} f_X(t)\,dt + \int_1^\infty f_X(t)\,dt = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}.$$

(c) We have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(X > 1)}{P(B)} = \frac{1}{3}\,\frac{12}{5} = \frac{4}{5}.$$

(d) Using the definition of expected value, we compute

$$\mathbb{E}(X) = \int_{-\infty}^\infty t\,f_X(t)\,dt = \int_{-2}^0 t\left(\frac{1}{6}t + \frac{1}{3}\right)\,dt + \int_0^2 t\frac{1}{3}dt = \frac{4}{9}.$$

2. Let X be a random variable. Suppose we are given the following information regarding its distribution :

$$P\big(\{X \le t^2\}\big) = 1 - \frac{1}{2}\,e^{-t^2},$$
$$P\big(\{|X| \le t\}\big) = 1 - \frac{1}{2}\,e^{-t} - \frac{1}{2}\,e^{-2\,t}.$$

(a) Compute the probability of the event $A = \{-1 \le X \le 2\}$.
(b) Determine $F_X(t)$, the cumulative distribution function of X.
(c) Determine $f_X(t)$, the probability density function of X.

Solution. (a) Notice that we can express A as,

$$A = \underbrace{(-1 \le X \le 1)}_{B} \cup \underbrace{(1 < X \le 2)}_{C}.$$

$P(B)$ can be computed easily as, $P(B) = 1 - \frac{1}{2}e^{-1} - \frac{1}{2}e^{-2}$. For $P(C)$, observe that

$$C \cup \underbrace{(X < 1)}_{D} = (X \le 2).$$

Since C and D are disjoint, we obtain

$$P(C) = P(X \le 2) - P(X < 1) = 1 - \frac{1}{2}e^{-2} - \left(1 - \frac{1}{2}e^{-1}\right) = \frac{1}{2}\left(e^{-1} - e^{-2}\right).$$

(b) Observe that if $t > 0$, $P(X \le t) = 1 - \frac{1}{2}e^{-t}$. If $t < 0$, notice that

$$(X < t) \cup (t \le X \le -t) = (X \le -t).$$

Therefore, since the two events on the left hand side are disjoint, we can write

$$P(X \le t) = 1 - \frac{1}{2}e^{t} - \left(1 - \frac{1}{2}e^{t} - \frac{1}{2}e^{2t}\right) = \frac{1}{2}e^{2t}.$$

Thus the cdf of X is,

$$F_X(t) = \begin{cases} \frac{1}{2}e^{2t}, & \text{if } t < 0, \\ 1 - \frac{1}{2}e^{-t}, & \text{if } 0 \le t. \end{cases}$$

(c) Differentiating the cdf, we find the pdf as,

$$f_X(t) = \begin{cases} e^{2t}, & \text{if } t < 0, \\ \frac{1}{2}e^{-t}, & \text{if } 0 \le t. \end{cases}$$

3. Let X be a random variable whose probability density function (pdf) is given by,

$$f_X(t) = \begin{cases} t/2, & \text{for } t \in [0,2], \\ 0, & \text{for } t \notin [0,2]. \end{cases}$$

Also, let A be the event $A = \{X \ge 1\}$.

(a) Determine $F_X(t) = P\big(\{X \le t\}\big)$, the cumulative distribution function (cdf) of X.

(b) Compute $\mathbb{E}(X)$, the expected value of X.

Solution. (a) Integrating the pdf, we find the cdf as,

$$F_X(t) = \int_{-\infty}^{t} f_X(s)\, ds = \begin{cases} 0, & \text{if } t < 0, \\ \frac{t^2}{4}, & \text{if } 0 \le t \le 2, \\ 1, & \text{if } 2 < t. \end{cases}$$

(b) We compute the expected value as,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} t\, f_X(t)\, dt = \int_0^2 t\frac{t}{2}\, dt = \frac{2}{3}.$$

4. Let X be a continuous random variable. Suppose we are given that,

$$P\{t_1 \le X \le t_2\} = \begin{cases} \frac{t_2 - t_1}{(t_1+2)(t_2+2)}, & \text{if } -1 \le t_1 \le t_2, \\ 0, & \text{if } \quad t_1 \le t_2 < -1. \end{cases}$$

(a) Find the probability of the event $\{X \ge 0\}$.

(b) Find the probability of the event $\{|X| \le 2\}$.

(c) Find $f_X(t)$, the probability density function (pdf) of X.

Solution. (a) Notice that

$$P(X \ge 0) = \lim_{t\to\infty} P(0 \le X \le t) = \lim_{t\to\infty} \frac{t}{2(t+2)} = \frac{1}{2}.$$

(b) Observe that $P(|X| \le 2) = P(X < -1) + P(-1 \le X \le 2) = 0 + \frac{3}{4} = \frac{3}{4}$.

(c) Observe that $P(X \le -1) = 0$. Therefore if $t > -1$, we have $P(X \le t) = P(-1 \le X \le t) = \frac{t+1}{t+2}$. From this, we obtain the cdf of X as,

$$F_X(t) = \begin{cases} 0, & \text{if } t < -1, \\ \frac{t+1}{t+2}, & \text{if } -1 \le t. \end{cases}$$

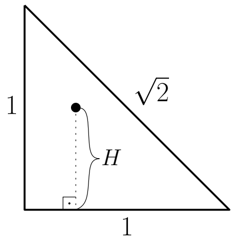
Differentiating the cdf, we obtain the pdf as

$$f_X(t) = \begin{cases} 0, & \text{if } t < -1, \\ \frac{1}{(t+2)^2}, & \text{if } -1 \le t. \end{cases}$$

MAT 271E – Homework 5

Due 06.04.2016

1. Consider the triangle below. Suppose we choose a point randomly inside the triangle (i.e. any point inside the triangle is equally likely to be chosen). Let H denote the vertical distance of the point to the base of the triangle.



(a) Specify $f_H(t)$, the probability distribution function (pdf) of H.

(b) Compute $\mathbb{E}(H)$, the expected value of H.

Solution. (a) Consider the cdf $F_H(t) = P(H \le t)$. Observe that, $0 \le H \le 1$ always. Therefore,

$$F_H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } 1 < t. \end{cases}$$

Finally for $0 \le t \le 1$, since the point is chosen randomly from the triangle, we have, $P(H \le t) = 1 - (1-t)^2$. Differentiating the cdf, we find the pdf as,

$$f_H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 2(1-t), & \text{if } 0 \le t \le 1, \\ 0, & \text{if } 1 < t. \end{cases}$$

(b) Using the definition of expected value, we can now compute,

$$\mathbb{E}(H) = \int t\, f_H(t)\, dt = \int_0^1 2t - 2t^2\, dt = \frac{1}{3}.$$

2. Suppose we break a unit length stick into two pieces randomly (i.e., we choose the break point randomly on the stick). Let X be the length of the shorter piece and Y be the length of the longer piece.

(a) Compute $\mathbb{E}(X)$.
(Hint : If you cannot compute this directly, do part (b) first.)

(b) Find the pdf of X.

(c) Find the pdf and compute the expected value of Y.

Solution. (a) Suppose Z is uniformly distributed on $[0, 1]$. Then, $X = \min(Z, 1 - Z)$. We can compute the expected value of X as,

$$\mathbb{E}(X) = \int \min(t, 1-t)\, f_Z(t)\, dt = \int_0^1 \min(t, 1-t)\, dt = \int_0^{1/2} t\, dt + \int_{1/2}^1 (1-t)\, dt = \frac{1}{8} + \frac{1}{8} = \frac{1}{4},$$

where we used the observation,

$$\min(t, 1-t) = \begin{cases} t, & \text{if } 0 \le t \le 1/2, \\ 1-t, & \text{if } 1/2 < t \le 1. \end{cases}$$

(b) We will make use of the cdf. Notice that for Z defined as in (a), the event $(X \le t)$ can be written equivalently as,

$$(\min(Z, 1-Z) \le t) = \underbrace{(Z \le t)}_{E_1} \cup \underbrace{(1 - Z \le t)}_{E_2}.$$

Notice that $E_2 = (1 - t \le Z)$. Recall that $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$. Observe,

$$P(E_1) = \begin{cases} 0, & \text{if } t < 0, \\ t, & \text{if } 0 \le t \le 1, \\ 1, & \text{if } 1 < t. \end{cases}$$

Check that $P(E_2) = P(E_1)$. Finally, observe that

$$P(E_1 \cap E_2) = P(1 - t \le Z \le t) = \begin{cases} 0, & \text{if } t < 1/2, \\ 2t - 1, & \text{if } 1/2 \le t \le 1, \\ 0, & \text{if } 1 < t. \end{cases}$$

Thus, we obtain the cdf of X as,

$$F_X(t) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \begin{cases} 0, & \text{if } t < 0, \\ 2t, & \text{if } 0 \le t \le 1/2, \\ 1, & \text{if } 1/2 < t. \end{cases}$$

Differentiating, we obtain the pdf as

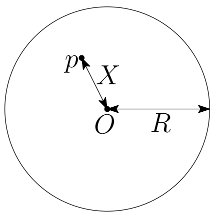
$$f_X(t) = \begin{cases} 0, & \text{if } t < 0, \\ 2, & \text{if } 0 \le t \le 1/2, \\ 0, & \text{if } 1/2 < t. \end{cases}$$

Note : An alternative is to observe that $E_1 \cup E_2 = \Omega - (E_1^c \cap E_2^c)$ and work with an intersection instead of a union like I did above. This in fact leads to a shorter derivation. Try it!

(c) Notice that $Y = 1 - X$. Therefore, $\mathbb{E}(Y) = \mathbb{E}(1 - X) = 1 - \mathbb{E}(X) = 3/4$. Also, $F_Y(t) = P(Y \le t) = P(1 - X \le t) = P(1 - t \le X) = 1 - P(X \le 1 - t) = 1 - F_X(1 - t)$. Differentiating, we find the pdf of Y as,

$$f_Y(t) = f_X(1 - t) = \begin{cases} 0, & \text{if } 1 - t < 0, \\ 2, & \text{if } 0 \le 1 - t \le 1/2, \\ 0, & \text{if } 1/2 < 1 - t \end{cases} = \begin{cases} 0, & \text{if } 1 < t, \\ 2, & \text{if } 1/2 \le t \le 1, \\ 0, & \text{if } t < 1/2. \end{cases}$$

3. Consider a disk with unknown radius R, centered around the origin O. Suppose your friend picks a point p on this disk randomly (i.e., any point is equally likely). Also, let the distance of p to the origin be denoted as X (see the figure).



(a) Find the probability of the event $\{X \le t\}$ (in terms of R) for $0 \le t \le R$.

(b) Find $f_X(t)$, the pdf of X.

Solution. (a) Notice that the ratio of the points inside the disk that are closer to the origin than t to the whole disk is given by t^2/R^2. Therefore, $P(X \le t) = t^2/R^2$.

(b) It follows from the observation in (a) that

$$F_X(t) = \begin{cases} 0, & \text{if } t < 0, \\ t^2/R^2, & \text{if } 0 \le t \le R, \\ 1, & \text{if } R < t. \end{cases}$$

Differentiating, we obtain the pdf as,

$$f_X(t) = \begin{cases} 0, & \text{if } t < 0, \\ 2t/R^2, & \text{if } 0 \le t \le R, \\ 0, & \text{if } R < t. \end{cases}$$

MAT 271E – Homework 6

Due 13.04.2016

1. Let X and Y be independent random variables. Assume that both X and Y are uniformly distributed on $[0,1]$.

   (a) Compute the probability of the event $A = \{X \leq Y\}$.

   (b) Given the event $A = \{X \leq Y\}$, compute the probability that '$X \leq 1/2$'.

   (c) Given the event $A = \{X \leq Y\}$, determine $F_{X|A}(t)$, the conditional cumulative distribution function (cdf) of X.

   (d) Given the event $A = \{X \leq Y\}$, find the conditional expectation of X.

   Solution. (a) Recall that

$$P(A) = \int P(A|Y=t)\, f_Y(t)\, dt = \int_0^1 P(A|Y=t)\, dt.$$

But for $0 \leq t \leq 1$, thanks to independence of X and Y, we have

$$P(A|Y=t) = P(X \leq Y|Y=t) = P(X \leq t|Y=t) = P(X \leq t) = t.$$

Therefore,

$$P(A) = \int_0^1 t\, dt = \frac{1}{2}.$$

(b) Let B be the event that $X \leq 1/2$. Recall that $P(B|A) = P(B \cap A)/P(A)$. Let $C = B \cap A$. Then,

$$P(C) = \int P(C|Y=t)\, f_Y(t)\, dt = \int_0^1 P(C|Y=t)\, dt.$$

But thanks to independence of X and Y, for $0 \leq t \leq 1$, we have

$$P(C|Y=t) = P\left((X \leq t) \cap (X \leq 1/2)|Y=t\right) = P\left((X \leq t) \cap (X \leq 1/2)\right) = \begin{cases} t, & \text{if } 0 \leq t \leq 1/2, \\ 1/2, & \text{if } 1/2 < t \leq 1. \end{cases}$$

Therefore,

$$P(C) = \int_0^1 P(C|Y=t)\, dt = \int_0^{1/2} t\, dt + \int_{1/2}^1 \frac{1}{2}\, dt = \frac{3}{8}.$$

It follows that $P(B|A) = P(C)/P(A) = 3/4$.

For $0 \leq s \leq 1$, let us define the event $C = (X \leq s) \cap A$. Observe that,

$$P(C|Y=t) = P\left((X \leq t) \cap (X \leq s)|Y=t\right) = P\left((X \leq t) \cap (X \leq s)\right) = \begin{cases} t, & \text{if } 0 \leq t \leq s, \\ s, & \text{if } s < t \leq 1. \end{cases}$$

Therefore,

$$P(C) = \int_0^1 P(C|Y=t)\, dt = \int_0^s t\, dt + \int_s^1 s\, dt = \frac{s^2}{2} + (1-s)\, s = s - \frac{s^2}{2}.$$

Thus,

$$F_{X|A}(s) = P(X \leq s|A) = \frac{P((X \leq s) \cap A)}{P(A)} = \begin{cases} 0, & \text{if } s < 0, \\ 2s - s^2, & \text{if } 0 \leq s \leq 1, \\ 1, & \text{if } 1 < s. \end{cases}$$

Notice that

$$f_{X|A}(t) = F'_{X|A}(t) = \begin{cases} 0, & \text{if } t < 0, \\ 2 - 2t, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{if } 1 < t. \end{cases}$$

Therefore,

$$\mathbb{E}(X|A) = \int f\, f_{X|A}(t)\, dt = \int_0^1 t\,(2-2t)\, dt = \frac{1}{3}.$$

2. Let X be a random variable, uniformly distributed on $[-1,1]$. Also, let $Y = \max\big(X, X^2\big)$.

   (a) Compute the probability of the event $A = \{Y \le 1/2\}$.

   (b) Find the cumulative distribution function (cdf) of Y.

   (c) Find $\mathbb{E}(Y)$, the expected value of Y.

   Solution. (a) Notice that

$$P(Y \le 1/2) = P(\max(X, X^2) \le 1/2) = P((X \le 1/2) \cap (X^2 \le 1/2))$$
$$= P((X \le 1/2) \cap (-1/\sqrt{2} \le X \le 1/\sqrt{2})) = P(-1/\sqrt{2} \le X \le 1/2) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}+1}{4\sqrt{2}}.$$

   (b) By a similar reasoning, we find that for $0 \le s \le 1$,

$$P(Y \le s) = P((X \le s) \cap (X^2 \le s)) = P(-\sqrt{s} \le X \le s) = \frac{1}{2}(s + \sqrt{s}).$$

   It follows that

$$F_Y(s) = P(Y \le s) = \begin{cases} 0, & \text{if } s < 0, \\ \frac{1}{2}(s+\sqrt{s}), & \text{if } 0 \le s \le 1, \\ 1, & \text{if } 1 < s. \end{cases}$$

   (c) Since Y is a function of X, we can use f_X. We have,

$$\mathbb{E}(Y) = \int \max(t, t^2)\, f_X(t)\, dt = \frac{1}{2}\int_{-1}^{1} \max(t,t^2)\, dt = \frac{1}{2}\left(\int_{-1}^{0} t^2\, dt + \int_0^1 t\, dt\right) = \frac{5}{12},$$

   where we used

$$\max(t, t^2) = \begin{cases} t^2, & \text{if } -1 \le t < 0, \\ t, & \text{if } 0 \le t \le 1. \end{cases}$$

3. Let Y be a random variable, uniformly distributed on $[0, X]$. Let Z be a random variable, uniformly distributed on $[0, X^2]$. Suppose that, given X, the random variables Y and Z are independent.

   (a) For $X = 1/2$, compute the probability of the event $\{Y \le Z\}$.

   (b) Assume that X is a random variable, uniformly distributed on $[0, 1]$. Compute the probability of the event $\{Y \le Z\}$.

   Solution. (a) For $X = 1/2$, Y is uniformly distributed on $[0, 1/2]$ and Z is uniformly distributed on $[0, 1/4]$. By the independence of Y and Z, we have,

$$P(Y \le Z) = \int P(Y \le Z | Z = t)\, f_Z(t) dt = \int_0^{1/4} P(Y \le t)\, 4\, dt = \int_0^{1/4} (2t)\, 4\, dt = \frac{1}{4}.$$

   (b) Given $X = s$ with $0 \le s \le 1$, Y is uniformly distributed on $[0, s]$ and Z is uniformly distributed on $[0, s^2]$. Noting that $s^2 \le s$, we then have,

$$P(Y \le Z | X = s) = \int P(Y \le Z | Z = t)\, f_Z(t) dt = \int_0^{s^2} P(Y \le t)\, \frac{1}{s^2}\, dt = \int_0^{s^2} \frac{t}{s}\frac{1}{s^2}\, dt = \frac{s}{2}.$$

   Therefore,

$$P(Y \le Z) = \int P(Y \le Z | X = s)\, f_X(s)\, ds = \int_0^1 \frac{s}{2}\, ds = \frac{1}{4}.$$

4. Let X be a discrete random variable with PMF,

$$P_X(k) = \begin{cases} 1/10, & \text{if } k \in \{1, 2, \ldots, 10\}, \\ 0, & \text{otherwise}. \end{cases}$$

   Also, let Y be a random variable with PMF,

$$P_Y(k) = \begin{cases} 1/X, & \text{if } k \in \{1, \ldots, X\}, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find the probability of the event $\{Y \leq 9\}$.

(b) Find the probability of the event $\{Y \leq 8\}$.

Solution. (a) Notice that if $(X \leq 9)$ occurs, then $(Y \leq 9)$ occurs with probability one. Observe also that the events $(X \leq 9)$ and $(X = 10)$ partition the sample space. Using these, we compute,

$$P(Y \leq 9) = P(Y \leq 9|X \leq 9)\, P(X \leq 9) + P(Y \leq 9|X = 10)\, P(X = 10) = 1\,\frac{9}{10} + \frac{9}{10}\,\frac{1}{10} = \frac{99}{100}.$$

(b) Reasoning similarly,

$$\begin{aligned} P(Y \leq 8) = P(Y \leq 8|X \leq 8)\, P(X \leq 8) + P(Y \leq 8|X = 9)\, P(X = 9) \\ + P(Y \leq 8|X = 10)\, P(X = 10) = 1\,\frac{8}{10} + \frac{8}{9}\,\frac{1}{10} + \frac{8}{10}\,\frac{1}{10}. \end{aligned}$$

5. Suppose we roll a fair die until we observe a 6. Assume that the tosses are independent. Let X be the total sum of the rolls.

(a) Find the probability that we roll n times, where $n \geq 1$.

(b) Let A_2 be the event that we roll twice. Compute $\mathbb{E}(X|A_2)$.

(c) Compute $\mathbb{E}(X)$.

Solution. (a) Observe that we roll n times if we do not observe a 6 for $n-1$ rolls, followed by a 6. Thanks to independence of the rolls, the probability of this event (denoted by A_n) is given by

$$P(A_n) = \left(\frac{5}{6}\right)^{n-1} \frac{1}{6}.$$

(b) Given that a certain roll is not a 6, the roll is uniformly distributed over the integers in $[1, 5]$. Therefore, its expected value is $\sum_{k=1}^{5} k\,\frac{1}{5} = 3$. Using this observation,

$$\mathbb{E}(X|A_2) = \mathbb{E}(\text{first roll}|A_2) + \mathbb{E}(\text{second roll}|A_2) = 3 + 6 = 9.$$

(c) Since the event A_1, A_2, ... partition the sample space, we have,

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{E}(X|A_n)\, P(A_n).$$

Noting that

$$\mathbb{E}(X|A_n) = 3n + 5,$$

we thus compute

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} (3n + 5) \left(\frac{5}{6}\right)^{n-1} \frac{1}{6} = 23.$$

MAT 271E – Homework 7

Due 11.05.2016

1. Let X be a random variable, uniformly distributed on $[0,1]$.Also, let Z be another random variable defined as $Z = X^\alpha$, where α is an unknown constant.

   (a) Compute $f_Z(t)$, the pdf of Z, in terms of α. (You can assume that $\alpha \neq 0$.)

   (b) Suppose we are given two independent observations of Z as z_1, z_2. Find the maximum likelihood (ML) estimate of α, in terms of z_1, z_2.

   (c) Evaluate the ML estimate you found in part (b) for $z_1 = e^{-3}$, $z_2 = e^{-4}$.

   Solution. (a) Note that $P(Z < 0) = 0$ and $P(Z > 1) = 0$. For $0 \leq t \leq 1$, we have $P(Z \leq t) = P(X^\alpha \leq t) = P(X \leq t^\alpha) = t^\alpha$. Differentiating, we find the pdf of Z as,

   $$f_Z(t) = \begin{cases} 0, & \text{if } t \notin [0,1], \\ \dfrac{1}{\alpha}\, t^{1/\alpha - 1}, & \text{if } t \in [0,1]. \end{cases}$$

   (b) The likelihood function for α is

   $$L(\alpha) = \frac{1}{\alpha^2}\,(z_1\, z_2)^{\,1/\alpha - 1}.$$

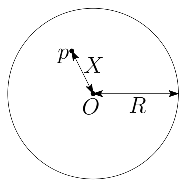
   Note that $z_1\, z_2 = e^c$ (for $c = \ln(z_1 z_2)$). Setting the derivative of $L(\alpha)$ to zero, we find that ML estimate should satisfy,

   $$L'(\alpha) = -\frac{2}{\alpha^3}\, e^{c/\alpha - c} - \frac{c}{\alpha^4} e^{c/\alpha - c} = 0.$$

   Solving this we find the ML estimate as, $\alpha_{\text{ML}} = -\dfrac{c}{2} = \dfrac{1}{2} \ln\left(\dfrac{1}{z_1\, z_2}\right)$.

   (c) Plugging in the values for z_1 and z_2, we find the ML estimate as $\alpha_{\text{ML}} = 7/2$.

2. Consider a disk with unknown radius R, centered around the origin O. Suppose your friend picks a point p on this disk randomly (i.e., any point is equally likely). Also, let the distance of p to the origin be denoted as X (see the figure).



   (a) Find the probability of the event $\{X \leq t\}$ (in terms of R) for $0 \leq t \leq R$.

   (b) Find $f_X(t)$, the pdf of X.

   (c) Consider $\hat{R} = 3X/2$ as an estimator for R. Is $\hat{R}$ biased or unbiased? If it is biased, propose an unbiased estimator for R. If it is unbiased, explain why.

   Solution. (a) For $0 \leq t \leq R$, we find $P(X \leq t)$ by computing the ratio of areas of a disk with radius t and R as $P(X \leq t) = \dfrac{\pi t^2}{\pi R^2}$.

   (b) Differentiating the cdf, we find,

   $$f_X(t) = \begin{cases} 0, & \text{if } t < 0, \\ \dfrac{2t}{R^2}, & \text{if } 0 \leq t \leq R, \\ 0, & \text{if } R < t. \end{cases}$$

   (c) We compute

   $$\mathbb{E}(\hat{R}) = \int \frac{3}{2} t\, f_X(t)\, dt = \int_0^R \frac{3}{2} t\, \frac{2t}{R}\, dt = R.$$

   The estimator is unbiased.
   (Something to think about : Consider the same question for a sphere instead of a disk)

3. Suppose X is a random variable, uniformly distributed on $[0,1]$. That is, the probability density function (pdf) of X is given by,

$$f_X(t) = \begin{cases} 1, & \text{if} \quad t \in [0,1], \\ 0, & \text{if} \quad t \notin [0,1]. \end{cases}$$

Let $Y = \theta X$, where θ is a constant such that $\theta > 2$.

(a) Find $f_Y(t)$, the pdf of Y (possibly in terms of θ).

(b) Compute the probability of the event $\{Y - 1 \le \theta \le Y + 1\}$ (possibly in terms of θ).

Solution. (a) Observe that if Y can take values in the range $[0,\theta]$. Now if $0 \le t \le \theta$, then we have $P(Y \le t) = P(X \le t/\theta) = t/\theta$. Differentiating this, we find,

$$f_Y(t) = \begin{cases} \frac{1}{\theta}, & \text{if} \quad t \in [0,\theta], \\ 0, & \text{if} \quad t \notin [0,\theta]. \end{cases}$$

(b) We rewrite the event as,

$$(Y-1 \le \theta \le Y+1) = (Y-1 \le \theta) \cap (\theta \le Y+1) = (Y \le \theta+1) \cap (\theta-1 \le Y) = (\theta-1 \le Y \le \theta+1).$$

Therefore,

$$P(Y-1 \le \theta \le Y+1) = P(\theta-1 \le Y \le \theta+1) = \int_{\theta-1}^{\theta+1} f_Y(t)\,dt = \int_{\theta-1}^{\theta} \frac{1}{\theta}\,dt = \frac{1}{\theta}.$$

4. Suppose X is a continuous random variable uniformly distributed on $[-1,1]$. Note that the pdf of X is given by

$$f_X(t) = \begin{cases} 1/2, & \text{if } -1 \le t \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

Also, let $Y = \theta X$, where θ is an unknown non-negative constant (so that $\theta = |\theta|$).

(a) Compute $\mathbb{E}(Y^2)$, and $\mathbb{E}(|Y|)$, possibly in terms of the unknown θ.

(b) Find some 'c' (possibly in terms of θ) such that $P\{|Y| - c \le \theta \le |Y| + c\} = 1/3$.

(c) Find an unbiased estimator for θ in terms of Y.

Solution. (a) We compute these expected values using the pdf of X.

$$\mathbb{E}(Y^2) = \theta^2\,\mathbb{E}(X^2) = \theta^2 \int_{-1}^{1} t^2\,\frac{1}{2}\,dt = \frac{\theta^2}{3}.$$

$$\mathbb{E}(|Y|) = \theta\,\mathbb{E}(|X|) = \theta \int_{-1}^{1} |t|\,\frac{1}{2}\,dt = \theta \int_{-1}^{0} -t\,\frac{1}{2}\,dt + \theta \int_{0}^{1} t\,\frac{1}{2}\,dt = \frac{\theta}{2}.$$

(b) We rewrite the event as,

$$(|Y|-c \le \theta \le |Y|+c) = (|Y|-c \le \theta) \cap (\theta \le |Y|+c) = (|Y| \le \theta+c) \cap (\theta-c \le |Y|) = (\theta-c \le |Y| \le \theta+c)$$

This can be expressed in terms of X as,

$$\{\theta - c \le |Y| \le \theta + c\} = \{1 - c/\theta \le |X| \le 1 + c/\theta\}.$$

Therefore,

$$P(|Y| - c \le \theta \le |Y| + c) = P(1 - c/\theta \le |X| \le 1 + c/\theta) = \frac{c}{\theta}$$

For this probability to be $1/3$, we must have $c = \theta/3$.

(c) By part (a), we know $\mathbb{E}(|Y|) = \theta/2$. Therefore an unbiased estimator for θ is $\hat{\theta} = 2|Y|$.