MAT 281E – Linear Algebra and Applications

Fall 2013

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Class Meets : 13.30 – 16.30, Friday
EEB 5202

Office Hours : 10.00 – 12.00, Friday

Textbook : G. Strang, 'Introduction to Linear Algebra', 4^{th} Edition, Wellesley Cambridge.

Grading : 2 Midterms (30% each), Final (40%).

Homework : There will be a homework almost every week but they will not be graded.

Webpage : http://ninova.itu.edu.tr/Ders/1039/Sinif/6402

Tentative Course Outline

- Solving Linear Equations via Elimination

  Linear system of equations, elimination, LU Decomposition, Inverses

- Vector Spaces

  The four fundamental subspaces, solving $Ax = b$, rank, dimension.

- Orthogonality

  Orthogonality, projection, least squares, Gram-Schmidt orthogonalization.

- Determinants

- Eigenvalues and Eigenvectors

  Eigenvalues, eigenvectors, diagonalization, application to difference equations, symmetric matrices, positive definite matrices, iterative splitting methods for solving linear systems, singular value decomposition.

MAT 281E – Homework 1

Due : 04.10.2013

1. Consider the matrices A and C given below

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad C = \begin{bmatrix} h & g & i \\ e & d & f \\ b & a & c \end{bmatrix}.$$

Notice that if we exchange the first and third rows of A, and then exchange the first and second columns of the resulting matrix, we obtain C. Find matrices P_1, P_2 such that

$$P_1 \, A \, P_2 = C.$$

Solution. Recall multiplying a matrix A on the left leads to row operations on A. Multiplying on the right leads to column operations. To exchange the first and third rows of A, multiply on the left by

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

To exchange the first and second columns, multiply on the right by

$$P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. Solve the linear system of equations

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 4 & 2 & 1 & 1 \\ 3 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 5 \\ -1 \end{bmatrix}$$

by Gaussian elimination. Use the augmented matrix for doing elimination. Also, write down the elimination matrix that you (implicitly) use at each elimination step.

Solution. We form the augmented matrix by augmenting the vector on the right hand side to the coefficient matrix :

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 & -6 \\ 4 & 2 & 1 & 1 & 3 \\ 3 & 1 & 1 & 0 & 5 \\ 2 & 2 & 0 & 1 & -1 \end{bmatrix}.$$

Let us now do elimination on the augmented matrix to reduce the coefficient matrix to an upper triangular matrix.

$$\begin{bmatrix} 2 & 1 & -1 & 3 & -6 \\ 4 & 2 & 1 & 1 & 3 \\ 3 & 1 & 1 & 0 & 5 \\ 2 & 2 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 2 & 1 & -1 & 3 & -6 \\ 0 & 0 & 3 & -5 & 15 \\ 3 & 1 & 1 & 0 & 5 \\ 2 & 2 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_3 - (3/2) r_1} \begin{bmatrix} 2 & 1 & -1 & 3 & -6 \\ 0 & 0 & 3 & -5 & 15 \\ 0 & -1/2 & 5/2 & -9/2 & 14 \\ 2 & 2 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{r_4 - r_1} \begin{bmatrix} 2 & 1 & -1 & 3 & -6 \\ 0 & 0 & 3 & -5 & 15 \\ 0 & -1/2 & 5/2 & -9/2 & 14 \\ 0 & 1 & 1 & -2 & 5 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 2 & 1 & -1 & 3 & -6 \\ 0 & -1/2 & 5/2 & -9/2 & 14 \\ 0 & 0 & 3 & -5 & 15 \\ 0 & 1 & 1 & -2 & 5 \end{bmatrix}$$

$$\xrightarrow{r_4 - (-2) r_2} \begin{bmatrix} 2 & 1 & -1 & 3 & -6 \\ 0 & -1/2 & 5/2 & -9/2 & 14 \\ 0 & 0 & 3 & -5 & 15 \\ 0 & 0 & 6 & -11 & 33 \end{bmatrix} \xrightarrow{r_4 - 2r_3} \underbrace{\begin{bmatrix} 2 & 1 & -1 & 3 & -6 \\ 0 & -1/2 & 5/2 & -9/2 & 14 \\ 0 & 0 & 3 & -5 & 15 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}}_{B}$$

The last matrix represents the system of equations

$$\begin{bmatrix}2 & 1 & -1 & 3\\ 0 & -1/2 & 5/2 & -9/2\\ 0 & 0 & 3 & -5\\ 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}-6\\ 14\\ 15\\ 3\end{bmatrix}.$$

We can solve for x_4 from the last equation as $x_4 = -3$. Substituting this value in the third equation,

$$3\,x_3 - 5(-3) = 15,$$

we obtain $x_3 = 0$. Using the values of x_3 and x_4 in the second equation,

$$(-1/2)x_2 + (5/2)\,0 + (-9/2)(-3) = 14,$$

we get $x_2 = -1$. Finally, from the first equation,

$$2x_1 + 1(-1) + (-1)(0) + (3)(-1) = -6,$$

we obtain $x_1 = -1$. Thus the solution is (Check it !)

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}-1\\ -1\\ 0\\ -3\end{bmatrix}.$$

Now, there are six elimination steps from A to B (count the number of arrows). Starting with the first these can be realized by multiplications on the left by the following matrices.

$$E_1=\begin{bmatrix}1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad E_2=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -3/2 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad E_3=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}$$

$$E_4=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad E_5=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\end{bmatrix},\quad E_6=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -2 & 1\end{bmatrix}.$$

Using these matrices, we can express the relation between A and B as,

$$E_6\,E_5\,E_4\,E_3\,E_2\,E_1\,A = B.$$

3. (a) Let I denote the $n \times n$ identity matrix and $a = \begin{bmatrix}a_1 & a_2 & \ldots & a_n\end{bmatrix}$ a length-n row vector. Consider the $(n+1) \times (n+1)$ matrix

$$B=\begin{bmatrix}1 & a\\ 0 & I\end{bmatrix}.$$

Here, 0 represents a zero vector of length-n. Find the inverse of B in terms of a.

(b) Let A be an $n \times n$ invertible matrix with inverse given as A^{-1}. Also, let a be as given in part (a). Consider the $(n+1) \times (n+1)$ matrix

$$C=\begin{bmatrix}1 & a\\ 0 & A\end{bmatrix},$$

constructed similarly as above. Find C^{-1}, the inverse of C.

Solution. (a) Consider a 2×2 matrix

$$\tilde{B}=\begin{bmatrix}1 & c\\ 0 & 1\end{bmatrix},$$

for some constant c. We can find the inverse of this matrix (either by inspection or Gauss-Jordan elimination), as

$$\tilde{B}^{-1}=\begin{bmatrix}1 & -c\\ 0 & 1\end{bmatrix}. \tag{1}$$

Based on this observation, treating the blocks of B as if they are scalars, one can suggest

$$B^{-1} = \begin{bmatrix} 1 & -a \\ 0 & I \end{bmatrix}.$$

Check that this is indeed the inverse of B (i.e. check that the blocks can be multiplied etc.).

(b) Suppose we multiply C by the block diagonal matrix,

$$\begin{bmatrix} 1 & 0 \\ 0 & A^{-1} \end{bmatrix},$$

where 0's represent blocks of zeros with possibly different sizes (what should the sizes be?). Notice that the product is,

$$\tilde{C} = \begin{bmatrix} 1 & a \\ 0 & I \end{bmatrix}.$$

We know the inverse of $\tilde{C}$ from part (a) (note that $B = \tilde{C}$). Therefore, we find the inverse of C as

$$C^{-1} = B^{-1} \begin{bmatrix} 1 & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} 1 & -a\,A^{-1} \\ 0 & A^{-1} \end{bmatrix}.$$

4. Recall that we defined the inner product of two length-n (column) vectors x, y as,

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i\, y_i.$$

Note that the inner product is linear in the sense that for a, b scalars and x, t vectors, we have

$$\langle a\,x + b\,t, y \rangle = a\,\langle x, y \rangle + b\,\langle t, y \rangle.$$

Now let A be a square matrix whose columns are denoted by c_i, i.e.,

$$A = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}.$$

Consider the inner product $\langle A\,x, y \rangle$. Find a square matrix B such that $\langle A\,x, y \rangle = \langle x, B\,y \rangle$, no matter how we choose x and y.
(Hint : All you need is the definition of the inner product and the linearity property above.)

Solution. Recall from block multiplication rules that if x is a length-n vector,

$$A\,x = x_1\,c_1 + x_2\,c_2 + \ldots + x_n\,c_n = \sum_{i=1}^{n} x_i\,c_i.$$

Therefore,

$$\begin{aligned} \langle A\,x, y \rangle &= \langle x_1\,c_1 + x_2\,c_2 + \ldots + x_n\,c_n, y \rangle \\ &= x_1\,\langle c_1, y \rangle + x_2\,\langle c_2, y \rangle + \ldots + x_n\,\langle c_n, y \rangle \\ &= \left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \underbrace{\begin{bmatrix} \langle c_1, y \rangle \\ \langle c_2, y \rangle \\ \vdots \\ \langle c_n, y \rangle \end{bmatrix}}_{z} \right\rangle. \end{aligned}$$

Observe that z is nothing but

$$z = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_n^T \end{bmatrix} y.$$

Therefore, $B = A^T$.

MAT 281E – Homework 2

11.10.2013

1. Suppose that a 3×3 matrix A whose rows are denoted by r_1, r_2 r_3, is invertible. Consider the matrix

$$B = \begin{bmatrix} 2r_1 - r_2 \\ 4r_1 + r_2 - r_3 \\ 6r_2 + r_3 \end{bmatrix}.$$

Is B invertible or not? Explain your reasoning.

Solution. Recall that if we multiply A on the left by some matrix C, then the rows of the product consist of linear combinations of the rows of A. Therefore, B can be expressed as,

$$B = \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 4 & 1 & -1 \\ 0 & 6 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}}_{A}.$$

We know that A is invertible. If C is also invertible, then B will be invertible with inverse given as $B^{-1} = A^{-1} C^{-1}$. However, if C is not invertible, B will not be invertible (why not?). Let us now check if C is invertible. We do so by doing elimination – all we need to see is whether the pivots are non-zero or not, that is we don't actuall need C^{-1}, so we don't work with the augmented matrix.

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & 1 & -1 \\ 0 & 6 & 1 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 6 & 1 \end{bmatrix} \xrightarrow{r_3 - 2r_2} \begin{bmatrix} \textcircled{2} & -1 & 0 \\ 0 & \textcircled{3} & -1 \\ 0 & 0 & \textcircled{3} \end{bmatrix}.$$

Note that the pivots (circled) are all non-zero. Therefore C is invertible. By the previous argument, B is invertible.

2. Suppose that an invertible matrix A has columns c_1, c_2, c_3. Suppose also that the matrices B and C are defined as,

$$B = \begin{bmatrix} (c_1 - c_2) & (c_3) & (2c_1 + c_2 - c_3) \end{bmatrix}, \quad C = \begin{bmatrix} (c_2 - c_3) & (c_1 + c_2 + c_3) & (3c_1 + c_3) \end{bmatrix}.$$

Here, the columns of the matrices are enclosed in parentheses. Find two matrices D, E such that $B = D\,C\,E$.

Solution. Recall that multiplication of A on the right leads to a product whose columns can be expressed as linear combinations of the columns of A. Thus, we have,

$$B = \underbrace{\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}}_{X}, \quad C = \underbrace{\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}}_{Z}$$

From the second equality, we get $A = C\,Z^{-1}$. Plugging this in the first equality, we obtain, $B = C\,Z^{-1}\,Y$. Therefore $B = D\,C\,E$ for $D = I$, $E = Z^{-1}\,Y$. To find $Z^{-1}\,Y$, do Gauss-Jordan elimination (work with the augmented matrix $\begin{bmatrix} Z & Y \end{bmatrix}$).

3. Find the LU decomposition of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{bmatrix}.$$

Solution. Let us do elimination on A,

$$A \xrightarrow[E_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[E_2]{r_3 - r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow[E_3]{r_3 + (2/3) r_2} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}.$$

The elimination matrices that (implicitly) realize these steps are,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}.$$

Therefore, we have that $E_3\, E_2\, E_1\, A = U$. Equivalently, $A = L\, U$ where $L = E_1^{-1}\, E_2^{-1}\, E_3^{-1}$. Note that the inverses of E_i are easy to obtain :

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}.$$

Multiplying these matrices we find L as,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2/3 & 1 \end{bmatrix}$$

MAT 281E – Homework 3

15.11.2013

1. Suppose we are given vectors $u_1, u_2, \ldots, u_k$ which form a basis for a space U and using them we define new vectors $z_1, \ldots, z_k$ as

$$\begin{aligned} z_1 &= u_1, \\ z_2 &= u_1 + u_2, \\ &\vdots \\ z_k &= \sum_{i=1}^{k} u_i. \end{aligned}$$

Does the sequence $z_1, \ldots, z_k$ also form a basis for U?

Solution. Notice that z_i's are related to u_i's through,

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ & & \ddots & & \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}}_{A} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}.$$

Here, A is a matrix composed of all zeros above the diagonal and one everywhere else. Note that A is invertible – to see this do elimination to find that all of the pivots are non-zero (I leave it to you to recognize the pattern in elimination – inverse of A has a very simple form). Invertibility of A implies that

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix} = A^{-1} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}$$

This equation implies that u_i's can be expressed as linear combinations of z_i's. This in turn means that z_i's also span U. To see this, note that for any $u \in U$, we can find a weights α_i such that

$$u = \alpha_1 \, u_1 + \alpha_2 \, u_2 + \ldots + \alpha_k \, u_k = \begin{bmatrix} u_1 & u_2 & \ldots & u_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix},$$

since u_i's span U. But we also have $u^T = z^T \, (A^{-1})^T$. Therefore,

$$u = \begin{bmatrix} z_1 & z_2 & \ldots & z_k \end{bmatrix} \underbrace{(A^{-1})^T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}}_{\beta} = \beta_1 \, z_1 + \beta_2 \, z_2 + \ldots + \beta_k \, z_k$$

Thus, any $u \in U$ can be expressed as a linear combination of z_i's. Thus, z_i's span U.

Now we need to see if z_i's are linearly independent. Suppose they are not. In that case, we can find weights α_i, not all zero such that

$$\alpha_1 \, z_1 + \alpha_2 \, z_2 + \ldots + \alpha_k \, z_k = \begin{bmatrix} z_1 & z_2 & \ldots & z_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = 0.$$

Using again the relation $z^T = u^T A^T$, we have,

$$\begin{bmatrix} u_1 & u_2 & \dots & u_k \end{bmatrix} A^T \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}}_{\gamma} = \gamma_1 u_1 + \gamma_2 u_2 + \ldots + \gamma_k u_k = 0.$$

Since A^T is invertible, we can conclude that not all γ_k's are equal to zero. That is, u_i's are linearly independent. But this is a clear contradiction because we already know that u_i's are linearly independent. Thus, the assumption leading to this contradiction must be false – z_i's are linearly independent.

Actually, the ongoing arguments also imply the following : In an n-dimensional space, any collection of n linearly independent vectors form a basis for the space.

2. Suppose U and V are two-dimensional subspaces of $\mathbb{R}^3$ and $U \neq V$. Show that there exists a vector $z \in \mathbb{R}^3$ such that $z \notin U$ and $z \notin V$.

   Solution. Since U and V are two dimensional spaces, we can find bases that consist of two vectors for each. Specifically, suppose $\{u_1, u_2\}$ is a basis for U and $\{s_1, s_2\}$ is a basis for V. Note that either u_1 or u_2 can be in V but not both. Because if both were in V, then u_1 and u_2, being independent, would also be a basis for V and we would have $U = V$, which is not the case. By reasoning similarly, we can say that one of $\{s_1, s_2\}$ has to be out of U. Suppose $u_1 \notin V$ and $s_1 \notin U$. Then $z = u_1 + s_1$ is in neither U nor V. To see this, note that if $z \in U$, then we can find constants α_i such that $z = \alpha_1 s_1 + \alpha_2 s_2$. But this means that $u_1 = (\alpha_1 - 1) s_1 + \alpha_2 s_2$, which implies that $u_1 \in V$, which is, by assumption, false. Thus, $z \notin V$. By a similar argument, it follows that $z \notin U$ either (modify the argument to show this on your own!).

3. Suppose U and V are two-dimensional subspaces of $\mathbb{R}^3$ and $U \neq V$.

   (a) Let Z be the intersection of U and V, i.e. $Z = U \cap V$. Is Z a subspace or not?

   (b) Let Z be the union of U and V, i.e. $Z = U \cup V$. Is Z a subspace or not?

   Solution. (a) It is a subspace. To see that, we need to check two conditions.

   (i) Suppose $u \in U \cap V$. Also, let α be a scalar. Since $u \in U$ and U is a space, $\alpha u \in U$ also. Repeating the same argument, since $u \in V$ and V is a space, $\alpha u \in V$ also. Thus, $\alpha u \in U \cap V$.

   (ii) Suppose both u and v are in $U \cap V$. Then, since u and v are both in U and U is a space, $u + v$ is also in U. Similarly, since u and v are both in V and V is a space, $u + v$ is also in V. To conclude, $u + v$ is in $U \cap V$.

   (b) It is not a subspace. Note that from Q2, we know that we can find $u \in U$, $v \in V$ such that $u + v$ is in neither U nor V. That is, although $u \in U \cup V$ and $v \in U \cup V$, $(u + v) \notin U \cup V$.

4. Let $v = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$ be a non-zero vector in $\mathbb{R}^3$ and consider the plane P defined as the solution of $v^T x = 0$. Note that P is a two dimensional subspace. Let s_1, s_2 be a basis for P. Show that the collection $\{s_1, s_2, v\}$ forms a basis for $\mathbb{R}^3$.

   Solution. Suppose that

   $$\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 v = 0, \tag{1}$$

   for some scalar α_i's. Note that in this case,

   $$u = \alpha_1 s_1 + \alpha_2 s_2 = -\alpha_3 v.$$

   But since u is a linear combination of s_1 and s_2, it is in P. Therefore it satisfies $v^T u = 0$. This is equivalent to,

   $$-v^T \alpha_3 v = -\alpha_3 (v_1^2 + v_2^2 + v_3^2) = 0.$$

   Since we know that v is a non-zero vector, we must have $\alpha_3 = 0$. But this means that $u = \alpha_1 s_1 + \alpha_2 s_2 = 0$. Since s_i's are linearly independent (recall that they form a basis for P), we must also have $\alpha_1 = \alpha_2 = 0$. Therefore, we showed that the only linear combination of s_1, s_2, v that gives the zero vector is the one with all weights (α_i's) equal to zero. Therefore, the collection s_1, s_2, v is linearly independent.

   This in turn means that the 3×3 matrix $A = \begin{bmatrix} s_1 & s_2 & v \end{bmatrix}$ is invertible. Thus, given an arbitrary $u \in \mathbb{R}^3$, we can solve $A x = u$ – that is we can represent u as a linear combination of the columns (i.e. s_1, s_2, v) of A. Thus they form a basis for $\mathbb{R}^3$.

   Alternatively, recalling the solution to Q1 above, we can argue that since $\mathbb{R}^3$ is a 3-dimensional space and s_1, s_2, v are 3 linearly independent vectors in $\mathbb{R}^3$, they form a basis for $\mathbb{R}^3$.

MAT 281E – Homework 4

Due 29.11.2013

1. Consider a plane P, described as the set of vectors $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ that satisfy the equation $x_1 - 2\,x_2 + 3\,x_3 = 0$. Find a basis for $P^{\perp}$, the orthogonal complement of P.

   Solution. Note that P is the nullspace of the matrix $A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$. We know from class that $N(A)^{\perp} = C(A^T)$. Therefore, $P = C(A^T)$. Since A^T contains a single vector, it actually forms a basis for $C(A^T)$.

2. Given an arbitrary b, we know that the system $A\,x = b$ might not have a solution, if $b \notin C(A)$. However, we noted in class that to find the best approximation to b, we can instead solve the system $A^T\,A\,x = A^T\,b$. Show that this system always has a solution.

   Solution. Note that we can decompose $b = b_1 + b_2$, where $b_1 \in N(A^T)$, $b_2 \in C(A)$. Since $b_2 \in C(A)$, we can find x such that $A\,x = b_2$. But since $A^T\,b_1 = 0$, we have, $A^T\,A\,x = A^T\,b_2 = A^T\,b_2 + A^T\,b_1 = A^T\,b$.

3. Consider a set of non-zero vectors as $\{q_1, \ldots, q_k\}$ such that $\langle q_i, q_j \rangle = 0$ for all (i, j) pairs with $i \neq j$ – that is the set of vectors are orthogonal. Show that, this implies that the vectors are also linearly independent. (Notice however that the converse does not hold – we can find a set of linearly independent vectors which are not orthogonal.)

   Solution. We showed this in class.

4. Find an orthonormal basis for the plane P in Question-1.

   Solution. Let us first find a vector from the nullspace of P. For this, recall that P is the nullspace of $A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$. Note that the second and the third columns are the free columns. Setting the second variable to one and the third variable to zero, we find a special solution as $s_1 = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T$. Since P is two dimensional, we need another vector for the basis. But the question asks that the basis be orthonormal. Therefore, the second basis vector should be orthogonal to s_1. In order to lie in the plane, it should also be orthogonal to the row space of A. Therefore, it can be obtained by finding the nullspace of

   $$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -6 \end{bmatrix}$$

   From this we have that $s_2 = \begin{bmatrix} -3/5 & 6/5 & 1 \end{bmatrix}$ is orthogonal to s_1 and lies in P. Normalizing, we obtain that $\{s_1/\|s_1\|, s_2/\|s_2\|\}$ is an orthonormal basis for P.

5. Consider a complex number of the form $z = z_r + i\,z_i$, where z_r is the real part and z_i is the imaginary part of this number. Note that we can also represent z with the length-two vector $\begin{bmatrix} z_r & z_i \end{bmatrix}^T$. Recall that we can also express z as

   $$z = \underbrace{\sqrt{z_r^2 + z_i^2}}_{|z|}\; e^{i\,\theta}$$

   where $\tan(\theta) = z_i/z_r$. Suppose we transform the vector $\begin{bmatrix} z_r & z_i \end{bmatrix}^T$ as,

   $$\begin{bmatrix} y_r \\ y_i \end{bmatrix} = A \begin{bmatrix} z_r \\ z_i \end{bmatrix}.$$

   Also, let $y = y_r + i\,y_i$.

   (a) Find a matrix A such that $y = z\,e^{i\alpha}$.

   (b) Find the inverse of A from part (a).

   Solution. (a) Recall Euler's relation : $e^{i\,\alpha} = \cos(\alpha) + i\sin(\alpha)$. Therefore,

   $$z\,e^{i\alpha} = (z_r + iz_i)\big(\cos(\alpha) + i\sin(\alpha)\big) = \underbrace{\Big(\cos(\alpha)\,z_r\; - \sin(\alpha)\,z_i\Big)}_{y_r} + i\,\underbrace{\Big(\sin(\alpha)\,z_r\; + \cos(\alpha)\,z_i\Big)}_{y_i}$$

Thus we can write

$$\begin{bmatrix} y_r \\ y_i \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}}_{A} \begin{bmatrix} z_r \\ z_i \end{bmatrix}.$$

Note that multiplying with A rotates the vector by α.

(b) Note that $z = e^{-i\alpha}\, y$. Therefore, if we replace α with $-\alpha$ in A, we should obtain A^{-1}. That is,

$$A^{-1} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

Observe that $A^{-1} = A^T$. Actually A is an orthogonal matrix. In general, rotation matrices are orthogonal.

MAT 281E – Homework 5

Due 06.12.2013

1. Consider an $n \times n$ matrix A which has ones on the antidiagonal and zero everywhere else. That is, A is of the form

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \vdots & & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

   (a) Find $|A|$ for $n = 2, 3, 4$.

   (b) Give a general expression of $|A|$ for a general n.

Solution. (a) For $n = 2$, A is,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Exchanging the rows we get the identity matrix. Since $|I| = 1$ and exchanging two rows has the effect of multiplying the determinant with -1, $|A| = -1$.

For $n = 3$, Let I_i, denote the rows of the identity matrix. Then, A is,

$$A = \begin{bmatrix} I_3 \\ I_2 \\ I_1 \end{bmatrix}.$$

Suppose we move I_1 up to the first row position in two steps, where at each step we exchange it with the row just above it. That is,

$$\begin{bmatrix} I_3 \\ I_2 \\ I_1 \end{bmatrix} \longrightarrow \begin{bmatrix} I_3 \\ I_1 \\ I_2 \end{bmatrix} \longrightarrow \begin{bmatrix} I_1 \\ I_3 \\ I_2 \end{bmatrix}.$$

Note that this preserves the order for the rest of the rows. Now exchange the second and third rows, to obtain I. This is not the fastest way to obtain I but it is systematic and the number of row exchanges is easy to count (which will be useful in the following). Overall, we did $2+1 = 3$ row exchanges, so $|A| = (-1)^3 = -1$.

You might guess that for $n = 4$, $|A| = -1$, but that would be wrong. I leave it to you to check that.

(b) Let I_i be defined as above. Then,

$$A = \begin{bmatrix} I_n \\ I_{n-1} \\ \vdots \\ I_2 \\ I_1 \end{bmatrix}.$$

Now suppose we move I_1 up as described above, without permuting the order of the rest of the rows. With $n - 1$ row exchanges, we reach the matrix

$$\begin{bmatrix} I_1 \\ I_n \\ I_{n-1} \\ \vdots \\ I_2 \end{bmatrix}.$$

Now do the same for I_2 on this modified matrix, this time placing it into the second row. With $n - 2$ row exchanges, we obtain

$$\begin{bmatrix} I_1 \\ I_2 \\ I_n \\ \vdots \\ I_3 \end{bmatrix}.$$

Continuing like, this, we obtain I by doing

$$(n-1)+(n-2)+\ldots+1=\frac{(n-1)\,n}{2}$$

row exchanges. Therefore,

$$|A|=(-1)^{n(n-1)/2}=\begin{cases}1, & \text{if } n \text{ or } (n-1) \text{ is divisible by } 4,\\ -1, & \text{otherwise.}\end{cases}$$

2. Consider the matrices

$$A=\begin{bmatrix}1 & 1 & 1\\ t & x & y\\ t^2 & x^2 & y^2\end{bmatrix},\quad B=\begin{bmatrix}1 & 1 & 1 & 1\\ t & x & y & z\\ t^2 & x^2 & y^2 & z^2\\ t^3 & x^3 & y^3 & z^3\end{bmatrix}.$$

   (a) Find an expression for the determinant of A.
   (Hint : Observe that the determinant will be a second order polynomial in terms of t. That is, $|A|$ is of the form

   $$|A|=c_2\,t^2+c_1\,t_1+c_0=c_2\,(t-z_0)\,(t-z_1),$$

   where z_i's are the roots of the quadratic polynomial. For which values of t is A singular? Those values should give z_i's.)

   (b) Give a condition in terms of x, y, z so that A is invertible.

   (c) Find an expression for the determinant of B.

Solution. (a) Observe that if $t=x$, then the first and the second colums of A are the same, in which case $|A|$ would be zero. Similarly, $|A|=0$ if $t=y$. In view of the hint, we have then $|A|=c_2\,(t-x)\,(t-y)$. Observe that c_2 is the coefficient of t^2 in the expression for $|A|$. But this is equal to $(y-x)$. Thus $|A|=(y-x)(t-x)(t-y)$.

(b) If the variables x, y, t are distinct (i.e., take different values), then the determinant is non-zero and A is invertible. Observe that this is also a necessary condition (what was the difference between 'necessity' and 'sufficiency'?), meaning that if A is invertible, then x, y, t must be distinct.

(c) Observe similarly that if $t=x$ or $t=y$ or $t=z$, then $|A|$ is zero. Thus,

$$|A|=c_3\,(t-x)\,(t-y)\,(t-z).$$

To determine c_3, observe that it is the coefficient of t_3 in the expression for $|A|$. Thus, making use of part (a),

$$c_3=-\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ x^2 & y^2 & z^2\end{vmatrix}=-(z-y)\,(x-y)\,(x-z).$$

3. Consider the matrix (notice the change in the matrix)

$$A=\begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 3 & 0\\ 0 & 3 & 2 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}.$$

Find the eigenvalues and eigenvectors of A.

Solution. Notice that $A-2I$ and $A-3I$ are singular, because both matrices have a zero row. Therefore 2 and 3 are eigenvalues. Recall that the associated eigenvectors can be found by finding vectors from the nullsapces of $A-2I$ and $A-3I$. In this case, they are easy to find : $\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$ and $\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T$. Consider now the submatrix

$$B=\begin{bmatrix}2 & 3\\ 3 & 2\end{bmatrix}.$$

Suppose that λ_1 and λ_2 are the eigenvalues of this matrix with associated eigenvectors x_1 and x_2. In that case, observe that

$$A \begin{bmatrix} 0 \\ x_i \\ 0 \end{bmatrix} = \lambda_i \underbrace{\begin{bmatrix} 0 \\ x_i \\ 0 \end{bmatrix}}_{s_i}, \qquad \text{for } i = 1, 2.$$

Therefore, s_i are eigenvectors of A with eigenvalues λ_i. To find λ_i and s_i, we go back to B, can compute the roots of $|B - \lambda I|$, which is

$$|B - \lambda I| = (2 - \lambda)(2 - \lambda) - 9 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).$$

Thus $\lambda_1 = 5$, $\lambda_2 = -1$. We find the associated eigenvectors as $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

4. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Suppose x is an eigenvalue of a matrix S with eigenvalue λ. Also, let U be a matrix related to S as $U = Q\,S\,Q^T$, where Q is an orthogonal matrix. Show that λ is an eigenvalue of U also. Can you find the corresponding eigenvector for U?

   (c) Find the eigenvalues and eigenvectors of B.

Solution. (a) As above, we observe that $A - I$ is singular, with $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in the nullspace. Consider now the submatrix

$$C = \begin{bmatrix} 4 & 2 \\ 2 & -1 \end{bmatrix}.$$

We find the eigenvalues by finding the roots of $|C - \lambda I|$. Once we obtain the eigenvalues λ_i, the eigenvectors x_i are found by computing vectors from the nullspace of $C - \lambda_i$. As in the question above, observe that

$$A \underbrace{\begin{bmatrix} 0 \\ x_i \end{bmatrix}}_{s_i} = \lambda_i\, s_i,$$

thus giving the eigenvalues.

(b) Since $S = Q^T\,U\,Q$ and $Sx = \lambda\,x$, we have, $U\,Q\,x = \lambda\,Q\,x$. Therefore $Q\,x$ is an eigenvalue of U with eigenvector λ.

(c) Notice that $B = P^T\,A\,P$, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus if λ_i are eigenvalues of A with eigenvectors e_i, then the same λ_i are also eigenvalues of B with eigenvectors $P\,e_i$.

MAT 281E – Linear Algebra and Applications, CRN : 10620

Midterm Examination – I

01.11.2013

Student Name : ______________________

Student Num. : ______________________

5 Questions, 120 Minutes
Please Show Your Work for Full Credit!

(20 pts) 1. Consider the system of linear equations

$$\begin{bmatrix} 2 & -1 & -1 \\ 4 & -1 & -3 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}.$$

(a) Find x_1, x_2, x_3 by Gaussian elimination.

(b) Write down the elimination matrix that you used in the first step of elimination.

(15 pts) 2. Consider the linear system of equations

$$\begin{bmatrix} a & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ b \end{bmatrix}.$$

(a) Find a pair (a, b) so that the system has a unique solution.

(b) Find a pair (a, b) so that the system has infinitely many solutions.

(c) Find a pair (a, b) so that the system has no solutions.

(15 pts) 3. Suppose A is a 3×3 matrix whose rows are denoted by r_1, r_2, r_3, that is, $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$. Also, let B be another 3×3 matrix given as

$$B = \begin{bmatrix} r_2 + 2\,r_3 \\ r_1 + r_2 \\ r_1 - 2\,r_2 \end{bmatrix}.$$

Suppose that for a specific vector c,

$$B \underbrace{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}_{c} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

Find $A\,c$.

(30 pts) 4. Consider the system of linear equations

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}}_{b}.$$

(a) Find a particular solution that solves this system of linear equations.

(b) Describe $N(A)$, the nullspace of A (that is, find the special solutions).

(c) What is the rank of A?

(d) Describe the whole solution set of the system of linear equations $A\,x = b$.

(20 pts) 5. Consider a plane P, in $\mathbb{R}^3$, described by the equation

$$x_1 + a_2\,x_2 + a_3\,x_3 = 0.$$

Suppose we are given two vectors u, v in P as,

$$u = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find a_2 and a_3.

(b) Find two vectors w, y that are not in P, such that $w \neq \alpha\,y$ for any real-valued α (that is, w cannot be obtained by multiplying y with a scalar.)

(c) Are the vectors u, v, w, y linearly indepdendent? Here, w and y are the vectors you found in part (b). Please explain your reasoning for full credit.

MAT 281E – Linear Algebra and Applications, CRN : 10620

Midterm Examination – II

13.12.2013

Student Name : ______________________

Student Num. : ______________________

5 Questions, 100 Minutes

Please Show Your Work for Full Credit!

(20 pts) 1. Let S be the set of vectors of the form

$$\begin{bmatrix}1\\2\\1\end{bmatrix} + \alpha_1 \begin{bmatrix}1\\1\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\\2\end{bmatrix},$$

where α_1 and α_2 are real numbers. Also let p be the vector

$$p = \begin{bmatrix}0\\4\\-2\end{bmatrix}.$$

Find the closest point of S to p.

(25 pts) 2. Let S be the solution set of

$$2\,x_1 + x_2 + x_3 = 0.$$

Also let V be the solution set of

$$x_1 + 2\,x_2 - x_3 = 0.$$

Notice that both S and V are subspaces of $\mathbb{R}^3$.

(a) Find a non-zero vector from $S^\perp$, the orthogonal complement of S.

(b) Let $U = S \cap V$. That is, U is the intersection of S and P. Find a non-zero vector from U.

(20 pts) 3. Let S be a plane in $\mathbb{R}^3$. Also, let the projection matrix onto S be given as

$$P = \begin{bmatrix}5/6 & -2/6 & 1/6\\-2/6 & 2/6 & 2/6\\1/6 & 2/6 & 5/6\end{bmatrix}.$$

Find a set of coefficients a_1, a_2, a_3, such that the solution set of the equation

$$a_1\,x_1 + a_2\,x_2 + a_3\,x_3 = 0$$

is equivalent to S.

(10 pts) 4. Compute the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

(25 pts) 5. Let A be a matrix with eigenvalues $\lambda_1 = -1/2$, $\lambda_2 = 3/4$, $\lambda_3 = 1$, where the associated eigenvectors are given as,

$$x_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Notice that the eigenvectors are orthogonal, but they are not normalized (i.e. $\|x_i\| \neq 1$).

(a) Compute $A^{-1} x_1$.

(b) Determine A^{-1}. (Check your answer!)

MAT 281E – Linear Algebra and Applications

Final Examination

07.01.2014

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work!

(20 pts) 1. Consider the system of equations

$$\underbrace{\begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 2 & 5 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}}_{b}.$$

(a) Describe the solution set of $A\,x = b$.

(b) Write down a basis for $N(A)$, the nullspace of A.

(c) What is the rank of A? What are the dimensions of the four fundamental subspaces, $N(A)$, $C(A)$, $N(A^T)$, $C(A^T)$?

(20 pts) 2. Consider the system of equations $A\,\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}.$$

Suppose that the solution set consists of all vectors of the form '$\mathbf{y} + \alpha_1\,\mathbf{u}_1 + \alpha_2\,\mathbf{u}_2$', where α_1, α_2 can be any real number and

$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Find a basis for $N(A)$, the nullspace of A.

(b) What are the dimensions of the four fundamental subspaces, $N(A)$, $C(A)$, $N(A^T)$, $C(A^T)$?

(c) Find a basis for $C(A^T)$, the row space of A.

(20 pts) 3. For an unknown set of coefficients a_1, a_2, a_3, let S be the solution set of

$$a_1\,x_1 + a_2\,x_2 + a_3\,x_3 = 0.$$

Also, let U be the solution set of

$$a_1\,x_1 + a_2\,x_2 + a_3\,x_3 = 1.$$

Notice that S is a subspace of $\mathbb{R}^3$ but U is not. Let P denote the projection matrix for S. Suppose we are given that

$$P\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2/3\\1/3\\-1/3\end{bmatrix}.$$

(a) Find a non-zero vector $s \in S$.

(b) Find a non-zero vector $v \in S^\perp$.

(c) Find a matrix A such that the nullspace of A is equivalent to S – that is, $N(A) = S$.

(d) Suppose $\begin{bmatrix}1\\1\\1\end{bmatrix} \in U$. Determine a_1, a_2, a_3.

(20 pts) 4. Suppose that S is a subspace of $\mathbb{R}^4$, spanned by the vectors s_1, s_2, where

$$s_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad s_2 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}.$$

(a) Find an orthonormal basis for S.

(b) Find two vectors $p \in S$ and $q \in S^\perp$ such that

$$p + q = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

(20 pts) 5. Let A be a matrix and x_1, x_2, column vectors, that satisfy the equations,

$$A\,x_1 = 4x_1 - 2x_2,$$
$$A\,x_2 = x_1 + x_2.$$

(a) Find the eigenvalues and the eigenvectors of the matrix B, given as,

$$B = \begin{bmatrix}4 & 1\\-2 & 1\end{bmatrix}.$$

(b) Find the eigenvectors of A and express the associated eigenvectors as linear combinations of x_1 and x_2. That is, if v is an eigenvector of A, find c, d such that $v = c\,x_1 + d\,x_2$.