TEL 502E – Detection and Estimation Theory

Spring 2016

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Class Meets : Tuesday, 9.30 – 12.30, EEB 5205

Textbook : 'Fundamentals of Statistical Signal Processing' (Vols. I,II), S. M. Kay, Prentice Hall.

Supplementary : 'An Introduction to Signal Detection and Estimation', H. V. Poor, Springer.
'Introduction to Probability', D. P. Bersekas, J. N. Tsitsiklis, Athena Scientific.
'Statistical Inference', G. Casella and R. L. Berger, Cengage Learning.
'Optimal Filtering', B. D. O. Anderson and J. B. Moore, Dover.

Webpage : There's a 'ninova' page, please log in and check.

Grading : Homeworks (10%), Midterm exam (40%), Final Exam (50%).

Tentative Course Outline

(1) Review of probability theory

(2) The estimation problem, minimum variance unbiased estimators

(3) The Cramér-Rao bound, sufficient statistics, Rao-Blackwell Theorem

(4) Best linear unbiased estimators maximum likelihood estimation

(5) Bayesian estimation, minimum mean square estimators, maximum a posteriori estimators

(6) The innovations process, Wiener filtering, recursive least squares, the Kalman filter

(7) Interval Estimation

(8) Simple Hypothesis Testing, the Neyman Pearson Lemma

(9) Bayesian tests, multiple hypothesis testing

(10) The matched filter, detection of stochastic signals

TEL502E – Homework 1

Due 01.03.2016

1. Suppose X_1, X_2 are independent variables uniformly distributed over $[0, \theta]$, where $\theta > 0$ is an unknown constant. In order to estimate θ, two estimators are proposed.

$$\theta_1 = X_1 + X_2, \quad \theta_2 = \max(X_1, X_2).$$

   (a) Determine whether θ_1 is unbiased or not. If it is biased, modify it to obtain an unbiased estimator.

   (b) Determine whether θ_2 is unbiased or not. If it is biased, modify it to obtain an unbiased estimator.

   (c) Compare the variances of the estimators you found in parts (a) and (b). Which would you prefer to use?

Solution. (a) Notice that $\mathbb{E}(X_1) = \mathbb{E}(X_2) = \theta/2$. Therefore, $\mathbb{E}(\theta_1) = \theta$ and θ_1 is unbiased.

(b) Let us find the pdf of θ_2 first. We will use the pdf for part (c) also. Note that the cdf of θ_2 is given as,

$$\begin{aligned}
F_{\theta_2}(t) &= P(\theta_2 \leq t) \\
&= P((X_1 \leq t) \cap (X_2 \leq t)) \\
&= P((X_1 \leq t))P((X_2 \leq t)) \\
&= \begin{cases} 0, & \text{if } t < 0, \\ t^2/\theta^2, & \text{if } 0 \leq t \leq \theta, \\ 1, & \text{if } \theta < t. \end{cases}
\end{aligned}$$

Differentiating, we find f_{θ_2} as,

$$f_{\theta_2}(t) = \begin{cases} 2t/\theta^2, & \text{if } 0 \leq t \leq \theta, \\ 0, & \text{otherwise.} \end{cases}$$

We now compute $\mathbb{E}(\theta_2)$ as,

$$\mathbb{E}(\theta_2) = \int_0^\theta t \, \frac{2t}{\theta^2} \, dt = \frac{2}{3} \, \theta.$$

Therefore θ_2 is biased, but $\tilde{\theta}_2 = \dfrac{3}{2}\theta_2$ is unbiased.

(c) First, note that by independence of X_1, X_2, we have,

$$\mathbb{E}(\theta_1^2) = \mathbb{E}(X_1)^2 + \mathbb{E}(X_1)^2 + 2\mathbb{E}(X_1)\,\mathbb{E}(X_2) = \frac{\theta^2}{3} + \frac{\theta^2}{3} + \frac{\theta^2}{2} = \frac{7}{6}\,\theta^2.$$

Therefore, $\operatorname{var}(\theta_1) = \dfrac{1}{6}\,\theta^2$.

Notice now that,

$$\mathbb{E}(\tilde{\theta}_2^2) = \frac{9}{4}\mathbb{E}(\theta_2^2) = \frac{9}{4}\int_0^\theta t^2 \, \frac{2t}{\theta^2} \, dt = \frac{9}{8}\theta^2.$$

Therefore $\operatorname{var}(\tilde{\theta}_2) = \dfrac{1}{8}\,\theta^2$.

Since

$$\operatorname{var}(\tilde{\theta}_2) < \operatorname{var}(\theta_1), \quad \text{for all } \theta,$$

I would prefer $\tilde{\theta}_2$.

2. Show that if $\operatorname{var}(X) = 0$ for a random variable, then $X = \mathbb{E}(X)$ (i.e., X is a constant).

Solution. This is an application of Chebyshev's inequality. But let us show it for a special case, using the Cauchy-Schwarz inequality (CSI).

Suppose $\operatorname{var} X = 0$ and X is a contiunous random variable with a pdf $f_X(t)$ that is non-zero in some interval I. Using the decomposition

$$t\, f_X(t) = \left[t\sqrt{f_X(t)}\right] \left[\sqrt{f_X(t)}\right],$$

we have, by CSI,

$$\Big(\mathbb{E}(X)\Big)^2 = \left[\int t f_X(t)\, dt\right]^2 \leq \left[\int t^2\, f_X(t)\, dt\right] \underbrace{\left[\int f_X(t)\, dt\right]}_{=1}.$$

Recall that, in order for this to hold with equality, we must have,

$$t\sqrt{f_X(t)} = c\sqrt{f_X(t)}, \qquad \text{for all } t,$$

where c is a non-zero constant. But since $f_X(t)$ is a non-zero function that integrates to 1, this is not possible (why?). Therefore, $\big[\mathbb{E}(X)\big]^2 < \mathbb{E}(X^2)$. Thus, $\operatorname{var}(X) = \mathbb{E}(X^2) - \big[\mathbb{E}(X)\big]^2 > 0$, which contradicts the assumption.

3. Suppose X is distributed as $\mathcal{N}(0, \sigma^2)$. Notice that X^2 is an unbiased estimator for σ^2. But suppose we are interested in σ and not σ^2. Is $|X|$ an unbiased estimator for σ?
(Hint : You do not need to evaluate $\mathbb{E}(|X|)$ to answer this question.)

Solution. Suppose $Z = |X|$ is an unbiased estimator of σ, that is, $\mathbb{E}(|X|) = \sigma$. Observe that $\mathbb{E}(Z^2) = \mathbb{E}(X^2) = \sigma^2$. Thus, $\operatorname{var}(Z) = \mathbb{E}(Z^2) - \big[\mathbb{E}(Z)\big]^2 = 0$. But this means, by Q2 that $Z = |X| = 0$, which is clearly not the case. Therefore, $\mathbb{E}(|X|) \neq \sigma$ and $|X|$ is biased as an estimator σ. In fact, we have $\mathbb{E}(|X|) < \sigma$ (why?).

4. Suppose X_1 and X_2 are independent random variables distributed as $\mathcal{N}(\sqrt{\theta}, \theta)$. Two estimators are proposed for θ :

$$\theta_1 = X_1\, X_2, \quad \theta_2 = X_1^2 + X_2^2.$$

(a) Determine whether θ_1 is unbiased or not. If it is biased, modify it to obtain an unbiased estimator.
(b) Determine whether θ_2 is unbiased or not. If it is biased, modify it to obtain an unbiased estimator.
(c) Compare the variances of the estimators you found in parts (a) and (b). Which would you prefer?
(Note : You might need the fourth moments of a Gaussian random variable for this part.)

Solution. (a) Observe that

$$\mathbb{E}(\theta_1) = \mathbb{E}(X_1\, X_2) = \mathbb{E}(X_1)\, \mathbb{E}(X_2) = \theta.$$

Thus θ_1 is unbiased.

(b) First notice that $\operatorname{var}(X_1) = \mathbb{E}(X_1^2) - \big(\mathbb{E}(X_1)\big)^2 = \theta$. Since $\mathbb{E}(X_1) = \sqrt{\theta}$, it follows that $\mathbb{E}(X_1^2) = 2\theta$. Similarly, $\mathbb{E}(X_2^2) = 2\theta$. Now,

$$\mathbb{E}(\theta_2) = \mathbb{E}(X_1^2 + X_2^2) = \mathbb{E}(X_1^2)\, \mathbb{E}(X_2^2) = 4\theta.$$

Thus θ_2 is biased, but we can derive an unbiased estimator from θ_2 as, $\tilde{\theta}_2 = \theta_2/4$.

(c) Notice that

$$\mathbb{E}(\theta_1^2) = \mathbb{E}(X_1^2)\, \mathbb{E}(X_2^2) = 4\theta^2.$$

Thus, $\operatorname{var}(\theta_1) = 3\theta^2$.
For the second estimator, we have (check this!),

$$\mathbb{E}(\tilde{\theta}_2^2) = \mathbb{E}\left(\frac{1}{16}(X_1^4 + X_2^4 + 2\, X_1^2\, X_2^2)\right) = \frac{7}{4}\,\theta^2.$$

Thus, $\operatorname{var}(\tilde{\theta}_2) = \dfrac{3}{4}\,\theta^2$. Observe that

$$\operatorname{var}(\tilde{\theta}_2) < \operatorname{var}(\theta_1), \quad \text{for all } \theta.$$

Thus, I would prefer $\tilde{\theta}_2$.

5. Suppose X_1 and X_2 are independent random variables distributed as $\mathcal{N}(2\theta, 1)$ and $\mathcal{N}(3\theta, 1)$ respectively, where θ is an unknown parameter.

   (a) Write down the joint pdf of X_1 and X_2.

   (b) Compute the Fisher information for θ, that is,

   $$I(\theta) = \mathbb{E}\left(\left[\partial_\theta\Big(\ln f(X_1, X_2; \theta)\Big)\right]^2\right),$$

   where $f(X_1, X_2; \theta)$ denotes the joint pdf of X_1 and X_2.

   (c) Find the UMVUE for θ in terms of X_1 and X_2.

Solution. (a) The joint pdf is,

$$f(x_1, x_2; \theta) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left((x_1 - 2\theta)^2 - (x_2 - 3\theta)^2\right)\right).$$

(b) Notice that

$$\partial_\theta\Big(\ln f(x_1, x_2; \theta)\Big) = 2(x_1 - 2\theta) + 3(x_2 - 3\theta) = 13\left(\frac{2x_1 + 3x_2}{13} - \theta\right).$$

Observe now that if $Z = (2X_1 + 3X_2)/13$, then $\mathbb{E}(Z) = \theta$. Thus, $I(\theta) = 13^2 \operatorname{var}(Z)$. But since X_1 and X_2 are independent, $\operatorname{var}(Z) = (2/13)^2 \operatorname{var}(X_1) + (3/13)^2 \operatorname{var}(X_2) = 1/13$. Thus $I(\theta) = 13$.

(c) In part (b), we found that,

$$\partial_\theta\Big(\ln f(x_1, x_2; \theta)\Big) = I(\theta)(z - \theta),$$

where $z = \dfrac{2x_1 + 3x_2}{13}$. Thus, the unbiased estimator $Z = \dfrac{2X_1 + 3X_2}{13}$ satisfies the CRLB and must be the UMVUE.

TEL502E – Homework 2

Due 08.03.2016

1. Consider a disk with an unknown radius r. We are interested in the area of the disk. For this, we measure the radius n times but each measurement contains some error. Specifically, suppose that the measurements are of the form $X_i = r + Z_i$ for $i = 1, 2, \ldots, n$, where Z_i's are independent zero-mean Gaussian random variables with known variance σ^2 (this is not a very good model for the error in this example but it is convenient to work with).

   (a) Find a sufficient statistic for r.

   (b) A professor suggests that we use

   $$\hat{A} = \pi \left(\frac{1}{n} \sum_{i=1}^{n} X_i^2 \right)$$

   as an estimator of the area. Determine if $\hat{A}$ is biased or not.

   (c) Find the UMVUE for the area of the disk.

   Solution. (a) Notice that $X_i \sim \mathcal{N}(r, \sigma^2)$. Thanks to independence, we find the joint pdf as,

   $$f_X(x_1, \ldots, x_n; r) = \frac{1}{(2\pi)^{n/2}\,\sigma^n} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - r)^2 \right).$$

   Observe that we can write this pdf as,

   $$f_X(t; r) = \left[\frac{1}{(2\pi)^{n/2}\,\sigma^n} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2 \right) \right] \left[\exp\left(-\frac{1}{2\sigma^2} \left(nr^2 + 2r \sum_{i=1}^{n} x_i \right) \right) \right].$$

   Thus by the factorization theorem, $T = \sum_i X_i$ is a sufficient statistic.

   (b) Notice that $\mathbb{E}(X_i^2) = \mathrm{var}(X_i) + \big(\mathbb{E}(X_i)\big)^2 = \sigma^2 + r^2$. Using this, we find $\mathbb{E}(\hat{A}) = \pi(\sigma^2 + r^2)$. Therefore, $\hat{A}$ is not a biased estimator of the area $A = \pi\, r^2$.

   (c) We found in part (a) that $T = \sum_i X_i$ is a sufficient statistic for r. If we set $A = \pi r^2$, it can also be shown that T is a sufficient statistic for A (using the factorization theorem). Assuming completeness, the Rao-Blackwell theorem suggests that the UMVUE is therefore a function of T. Consider T^2. Observe that

   $$\mathbb{E}(T^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbb{E}(X_i\, X_j) = n\sigma^2 + nr^2 + n\,(n-1)\, r^2 = n\sigma^2 + n^2 r^2.$$

   Therefore, $\tilde{A} = \pi\,(T^2 - n\sigma^2)/n^2$ is an unbiased estimator of the area which is a function of the sufficient statistic for the area. Therefore it must be the UMVUE we are looking for, by the Rao-Blackwell theorem.

2. Suppose X_1, X_2 are independent random variables distributed as $\mathcal{N}(0, \theta)$, $\mathcal{N}(0, 2\theta)$, where θ is an unknown positive constant.

   (a) Find a sufficient statistic for θ.

   (b) Find the UMVUE for θ.

   Solution. (a) The joint pdf is,

   $$f_X(x_1, x_2) = \frac{1}{2\sqrt{2}\,\pi\,\theta} \exp\left(-\frac{2X_1^2 + X_2^2}{4\theta} \right).$$

   Thus, $T = 2X_1^2 + X_2^2$ is a sufficient statistic, by the factorization theorem.

   (b) Observe that $\mathbb{E}(T) = 4\theta$. Thus, assuming that T is complete, $\hat{\theta} = T/4$ is the UMVUE by the Rao-Blackwell theorem.

3. Suppose X_1, X_2, X_3 are independent random variables and the pdf of X_k is given as,

$$f_k(t) = \begin{cases} \frac{1}{k\theta} \exp\left(-\frac{t}{k\theta}\right), & \text{if } 0 \leq t, \\ 0, & \text{if } t \leq 0, \end{cases}$$

for $k = 1, 2, 3$, where θ is a positive unknown.

(a) Find a sufficient statistic for θ and compute its expected value.

(b) Find a function of the sufficient statistic which is unbiased as an estimator of θ.

(Note : $\int_0^\infty t\, e^{-t}\, dt = 1$.)

Solution. (a) Note that the joint pdf is given as,

$$f_X(x;\theta) = [u(x_1)\, u(x_2)\, u(x_3)] \left[\frac{1}{6\theta^3} \exp\left(-\frac{1}{6\theta}\,(6x_1 + 3x_2 + 2x_3)\right)\right],$$

where u denotes the unit step function. Thus $T = (6x_1 + 3x_2 + 2x_3)$ is a sufficient statistic for θ. Observe that

$$\mathbb{E}(X_k) = \int_0^\infty \frac{x}{k\theta} \exp\left(-\frac{x}{k\theta}\right)\, dt = k\theta \int_0^\infty s \exp\left(-s\right)\, ds = k\theta.$$

Therefore, $\mathbb{E}(T) = 18\theta$.

(b) It follows by the previous discussion that $\hat{\theta} = \theta/18$ is such an estimator.

TEL502E – Homework 3

Due 22.03.2016

1. Consider a discrete random variable X whose probability mass function (pmf) depends on a parameter θ, where $\theta \in \{0, 1, 2\}$. Suppose that X takes values in $\{0, 1, 2, 3\}$ and its pmf for different values of θ, denoted by $P(x|\theta)$, is as given below.

x	$P(x\|\theta = 0)$	$P(x\|\theta = 1)$	$P(x\|\theta = 2)$
0	1/8	1/4	0
1	1/4	1/2	1/3
2	3/8	1/8	1/3
3	1/4	1/8	1/3

(a) Suppose we are given a realization of X as $x = 1$. Find the maximum likelihood estimate (MLE) for θ.

(b) Suppose we are given two independent realizations of X as $x_1 = 1$, $x_2 = 2$. Find the MLE for θ.

Solution. (a) Note that the likelihood function in this case is $L(\theta) = P(1|\theta)$. According to the table, $L(\theta)$ is maximized for $\hat{\theta} = 1$ (where $L(\hat{\theta}) = 1/2$).

(b) Thanks to independence, the likelihood function is given as $L(\theta) = P(1|\theta)\, P(2|\theta)$. We then have $L(0) = 3/32$, $L(1) = 1/16$, $L(2) = 1/9$. Thus, the ML estimate is $\hat{\theta} = 2$.

2. Suppose X_1, X_2, ..., X_n are independent and identically distributed random variables with pdf

$$f_{X_i}(t) = \begin{cases} 0, & \text{if } t < 0, \\ \theta^{-t} \ln(\theta), & \text{if } t \geq 0, \end{cases}$$

where $\theta > 1$ is an unknown constant.

(a) Find the maximum likelihood estimator for θ in terms of X_1, X_2, ..., X_n.

(b) Specify whether the estimator you found is biased or not.
(Hint : $\int_0^\infty x\, c^{-x}\, dx = \big(\ln(c)\big)^2$, if $c > 1$.)

Solution. (a) Given $X_i = x_i > 0$, the likelihood function is given as,

$$L(\theta) = \big(\ln(\theta)\big)^n \, \theta^{-\left(\sum_i x_i\right)}.$$

Setting the derivative of the log-likelihood function to zero, we find that the maximizer of this expression satisfies,

$$\frac{n}{\ln(\theta)} \frac{1}{\theta} - \left(\sum_i x_i\right) \theta = 0.$$

Solving for θ, we find the ML estimate as $\exp(n/\sum_i x_i)$. Therefore, the ML estimator is,

$$\hat{\theta} = \exp\left(\frac{n}{\sum_i X_i}\right).$$

(b) First notice that, by the provided hint, $\mathbb{E}(X_i) = 1/\ln(\theta)$. Therefore,

$$\mathbb{E}\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{1}{\ln(\theta)}.$$

Recall that Jensen's inequality states that if f is a strictly convex function and X is a continuous random variable, then

$$f\left(\mathbb{E}(X)\right) < \mathbb{E}\big(f(X)\big).$$

Observe that for $t > 0$, $f(t) = \exp(1/t)$ is a strictly convex function. Therefore, it follows that

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}\left(f\left(\frac{\sum_{i=1}^n X_i}{n}\right)\right) > f\left(\mathbb{E}(X)\right) = \theta.$$

Therefore, $\hat{\theta}$ is a biased estimator.

3. Let X_1, X_2 be independent Gaussian random variables with mean θ and variance 1. Also, let θ be a random variable uniformly distributed on $[0,1]$ – that is, the pdf of θ is given by,

$$f_\theta(t) = \begin{cases} 1, & \text{if } t \in [0,1], \\ 0, & \text{if } t \notin [0,1]. \end{cases}$$

(a) Find the joint pdf of θ, X_1, X_2. That is, find $f_{\theta,X_1,X_2}(t, x_1, x_2)$.

(b) Find the maximum a posteriori (MAP) estimate of θ.

(c) Evaluate the estimator you found in part (b) if the data is as given below.

(i) $x_1 = 3/4$, $x_2 = 1$.

(ii) $x_1 = 1/2$, $x_2 = 2$.

Solution. (a) The joint pdf is given as,

$$\begin{aligned} f_{X_1,X_2,\Theta}(x_1,x_2,t) &= f_{X_1,X_2|\Theta}(x_1,x_2|t)\, f_\theta(t) \\ &= \begin{cases} \dfrac{1}{2\pi} \exp\left(-\dfrac{(x_1-t)^2 + (x_2-t)^2}{2}\right), & \text{if } 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \dfrac{1}{2\pi} \exp\left(-\dfrac{x_1^2 + x_2^2}{2}\right) \cdot \exp\left(\dfrac{x_1+x_2}{2} t - t^2\right), & \text{if } 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(b) Notice that for fixed x_1, x_2, we need to maximize the term $\exp\left(\dfrac{x_1+x_2}{2}t - t^2\right)$ subject to $t \in [0,1]$. This is equivalent to minimizing $t^2 - \dfrac{x_1+x_2}{2}t$ with respect to $t \in [0,1]$. But this is a quadratic with a minimum at $(x_1+x_2)/2$. Therefore, the MAP estimate is given as,

$$\hat{t} = \begin{cases} 0, & \text{if } \dfrac{x_1+x_2}{2} < 0, \\ \dfrac{x_1+x_2}{2}, & \text{if } 0 \le \dfrac{x_1+x_2}{2} \le 1, \\ 1, & \text{if } 1 < \dfrac{x_1+x_2}{2}. \end{cases}$$

(c) (i) $\hat{t} = 7/8$. (ii) $\hat{t} = 1$.

4. Suppose we observe $X = \theta + Z$, where θ and Z are independent random variables. Suppose also that θ is uniformly distributed over the unit interval and Z is a standard normal random variable (i.e., $\mathcal{N}(0,1)$). That is, the pdfs of θ and Z are,

$$\begin{aligned} f_\theta(t) &= u(t)\, u(1-t), \\ f_Z(z) &= \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \end{aligned}$$

where u denotes the step function.

(a) Find the joint pdf of X and θ, that is, $f_{X,\theta}(x,t)$.

(b) Find the maximum a posteriori (MAP) estimator for θ in terms of X.

(c) Evaluate the estimator you found in part (b) if the observation is given as

(c.1) $x = 1/4$,

(c.2) $x = -1$,

(c.3) $x = 2$.

Solution. (a) Notice that $f_{X|\theta}(x|t) = f_Z(x-t)$. Therefore, the joint pdf of X and θ is obtained as,

$$f_{X,\theta}(x,t) = f_{X|\theta}(x|t)\, f_\theta(t) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{1}{2}(x-t)^2\right), & \text{if } 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) For fixed x, the joint pdf is maximized for

$$\hat{t} = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \le x \le 1, \\ 1, & \text{if } 1 < x. \end{cases}$$

(c) Evaluating the estimator, we find, (c.1) $\hat{t} = 1/4$, (c.2) $\hat{t} = 0$, (c.3) $\hat{t} = 1$.

5. Suppose X is a Gaussian random variable with mean θ and variance 1. Suppose θ is also a Gaussian random variable with mean 2 and variance 3.

(a) Find the pdf of X.

(b) Find the minimum mean square estimate (MMSE) of θ given X.

Solution. (a) Notice that X can be written as the sum of a standard normal random variable Z and θ, where Z and θ are independent. Since the sum of Gaussian random variables are Gaussian, X is Gaussian. Therefore, it suffices to find the mean and the variance of X. But $\mathbb{E}(X) = \mathbb{E}(Z + \theta) = 2$. Also, since Z and θ are independent, we have, $\text{var}(X) = \text{var}(Z) + \text{var}(\theta) = 4$. Thus,

$$f_X(x) = \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{1}{8}(x-2)^2\right).$$

(b) Notice that the joint pdf of X and θ is given as,

$$f_{X,\theta}(x,t) = f_{X|\theta}(x|t)\, f_\theta(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\,(x-t)^2\right) \frac{1}{\sqrt{6\pi}} \exp\left(-\frac{1}{6}\,(t-2)^2\right).$$

We find,

$$\begin{aligned} f_{\theta|X}(t|x) &= \frac{f_{X,\theta}(x,\theta)}{f_X(x)} \\ &= c \exp\left(\underbrace{-\frac{1}{2}(x-t)^2 - \frac{1}{6}(t-2)^2 + \frac{1}{8}(x-2)^2}_{h(t)}\right), \end{aligned}$$

where c is a constant. Notice that the form of $f_{\theta|X}(t|x)$, for fixed x and variable t, is the same as that of a Gaussian random variable. To find the mean and variance of this random variable, it's sufficient to find the maximum of $h(t)$ and the factor that multiplies t. But note that, since we are interested in $\mathbb{E}(\theta|X)$, finding the mean is sufficient for our purposes. The mean can be found by setting the derivative of h to zero. This gives the equation,

$$-(t-x) - \frac{1}{3}(t-2) = 0.$$

Solving for t, we find $t = 3/4(x + 2/3)$. Thus,

$$\mathbb{E}(\theta|X) = \frac{3}{4}X + \frac{1}{2} = \frac{3}{4}(X-2) + 2.$$

TEL 502E – Detection and Estimation Theory

Midterm Examination

29.03.2016

4 Questions, 90 Minutes

Please Show Your Work for Full Credit!

(25 pts) 1. Suppose $X_1, X_2, \ldots, X_n$ are independent Gaussian random variables with unit variance ($\sigma^2 = 1$) and means $\mathbb{E}(X_k) = k\,\theta$, where θ is an unknown constant.

(a) Compute the Cramer-Rao lower bound for θ.

(b) Find the uniformly minimum variance unbiased estimator (UMVUE) for θ.

(25 pts) 2. Suppose X_1, X_2 are independent identically distributed random variables, and each one is distributed according to the pdf

$$f_X(x) = \begin{cases} \dfrac{2}{\theta^2}\, x, & \text{if } 0 \leq x \leq \theta, \\ 0, & \text{otherwise,} \end{cases}$$

where θ is an unknown constant.

(a) Find a sufficient statistic T for θ. Evaluate $\mathbb{E}(T)$ for the statistic you found.

(b) Find a function of T, that is $g(T)$, such that $\mathbb{E}\big(g(T)\big) = \theta$.

(25 pts) 3. Suppose that given θ, X is a random variable with pdf

$$f_X(x) = \begin{cases} \exp(\theta - x), & \text{if } x \geq \theta, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose further that θ is itself a random variable with pdf

$$f_\theta(t) = \begin{cases} \exp(-t), & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Write down the joint pdf of X and θ.
(Note : Pay attention to the support of the pdfs. You can use the step function $u(\cdot)$ to find a neat expression.)

(b) Find the marginal pdf of X.

(c) Find the minimum mean squared error (MMSE) estimator for θ in terms of X.

(25 pts) 4. Suppose that given θ, the random variables $X_1, X_2, \ldots, X_n$ are independent and identically distributed according to the pdf

$$f_X(x) = \begin{cases} \theta \exp(-\theta x), & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose θ is also a random variable with pdf

$$f_\theta(t) = \begin{cases} \exp(-t), & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Write down the joint pdf of $X_1, \ldots X_n$ and θ.

(b) Find the maximum a posteriori (MAP) estimator for θ in terms of $X_1, \ldots X_n$.

(c) Evaluate the estimator you found above for,

   (i) $X_1 = 1$, $X_2 = 2$.

   (ii) $X_1 = 3$, $X_2 = 2$, $X_3 = 1$.

TEL 502E – Detection and Estimation Theory

Final Examination

27.06.2016

Student Name : ______________________

Student Num. : ______________________

4 Questions, 100 Minutes

Please Show Your Work for Full Credit!

(25 pts) 1. Suppose Z_1, Z_2, are independent and identically distributed random variables with pdf

$$f_{Z_i}(t) = \begin{cases} e^{-t}, & \text{if } 0 \le t, \\ 0, & \text{if } t < 0. \end{cases}$$

Suppose we observe $X_i = Z_i + \theta$, where θ is an unknown parameter, for $i = 1, 2$.

(a) Find a sufficient statistic T for θ (which is a function of X_1, X_2).

(b) Evaluate $\mathbb{E}(T)$ for the statistic you found in (a).
(Hint: $\int_0^\infty x\, e^{-sx}\, dx = 1/s^2$ for $s > 0$.)

(c) Find an unbiased estimator for θ in terms of X_1, X_2.

(d) Find the UMVUE for θ in terms of X_1, X_2.

(25 pts) 2. Suppose we are given a biased coin with $P(\text{Head}) = p$, where p is unknown.

(a) In order to estimate p, we toss the coin 5 times and observe the sequence $S = (HHTHT)$. What is the maximum likelihood estimate of p given these observations?

(b) If we had enough time to toss the coin n times, what would be the ML estimate?

(c) Is the ML estimator above unbiased?

(25 pts) 3. Suppose X_1 and X_2 are independent Gaussian random variables. Suppose also that $X_1 \sim \mathcal{N}(1, 2)$, and $X_2 \sim \mathcal{N}(3, 4)$. Also let $Z = X_1 + X_2$.

(a) Find the pdf of Z.

(b) Find the minimum mean square error (MMSE) estimator for X_1 in terms of Z.

(25 pts) 4. Suppose you have access to a random number generator that produces independent standard Gaussian random variables. After obtaining the realizations of three random variables say x_1, x_2, x_3, you supply two numbers z_1, z_2 to your friend. In order to form z_1, z_2, you either set $(z_1, z_2) = (x_1 + x_2, x_3)$, or set $(z_1, z_2) = (x_1, x_2 + x_3)$. Your friend tries to guess how the numbers z_1, z_2 were formed. Suppose the null hypothesis H_0 is $(z_1, z_2) = (x_1 + x_2, x_3)$, and the alternative hypothesis H_1 is $(z_1, z_2) = (x_1, x_2 + x_3)$.

(a) Find a test statistic T such that the Neyman-Pearson test is of the form,

$$\begin{cases} T(z_1, z_2) > \gamma & \Rightarrow \text{decide } H_0, \\ T(z_1, z_2) < \gamma & \Rightarrow \text{decide } H_1. \end{cases}$$

(b) For the test above, find the constant γ such that $P(\text{Type-I error}) = \alpha$. You can use the Q function defined as

$$Q(s) = \int_s^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) dt.$$